

Double real radiation corrections to $t\bar{t}$ production at the LHC: the $gg \rightarrow t\bar{t}q\bar{q}$ channel

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ABSTRACT: We present the double real radiation contributions to the $t\bar{t}$ hadronic production cross section stemming from the partonic process $gg \rightarrow t\bar{t}q\bar{q}$. We explicitly construct the antenna subtraction terms for this gluon-gluon initiated process starting from the double soft behaviour of the double real radiation matrix elements using soft currents. Those subtraction terms, given in leading and subleading colour contributions, require the use of new genuine NNLO four-parton antenna functions involving massive fermions. Those are also presented together with their infrared limits in this paper. We checked the validity of our subtraction terms numerically by showing that the ratio between the real radiation matrix elements and the subtraction terms approaches unity in all single and double unresolved regions of phase space.

KEYWORDS: QCD, Jets, Collider Physics, NLO and NNLO calculations with massive particles.

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1. Introduction

With a mass $m_t = 173 \pm 1.3$ GeV, the top quark is the heaviest quark produced at colliders and due to its very large mass it decays before it hadronises. Top quarks [1, 2] are measured through their decay into a bottom quark and a subsequently decaying W -boson, yielding up to six-jet final states for top quark pair production. Meaningful searches for these signals require not only a very good anticipation of the expected signal, but also of all Standard Model backgrounds yielding identical final state signatures.

By studying the properties of top-quarks in detail, it is hoped to elucidate the origin of particle masses and the mechanism of electroweak symmetry breaking. Since its discovery at the Fermilab Tevatron, a number of its properties (mass, couplings) have been determined to an accuracy of ten to twenty per cent. With the large number of top quark pairs expected to be produced at the LHC [3], the study of its properties will become precision physics. In particular the $t\bar{t}$ production cross section is expected to be measured with an accuracy of the order of five percent. These very precise measurements have to be matched unto equally accurate theoretical predictions. Fixed-order calculations of these observables need to be considered at least at the next-to-leading (NLO), if not even at next-to-next-to-leading order (NNLO) in perturbative QCD.

For any hadronic observable, a theoretical prediction is obtained at a given order in α_s when all partonic channels contributing at that order to the partonic cross section are summed and convoluted with the appropriate parton distribution functions. Beyond leading order, these partonic channels contain both ultraviolet and infrared (soft and collinear) singularities. The ultraviolet poles are removed by renormalisation, while collinear poles originating from the radiation of initial-state partons are cancelled by mass factorisation counterterms. The remaining soft and collinear infrared poles cancel among each other only when all partonic channels are summed over [4, 5].

NLO predictions for top quark pair production cross sections have been known already for some time [6–8]. Next-to-leading-logarithmic resummation (NLL) are also known [9–11], and even the NNLL resummation effects have been completed in [12]. These predictions lead to a theoretical uncertainty of the order of ten per cent. The same precision is available

for single top quark production [13], top-pair-plus-jets production [14–16] and for top-pair-plus-bottom-pair production [17, 18].

The top quark appears as virtual particle in hadron collider processes and due to the small ratio between the top quark width and its mass, it is possible to factor the cross section of processes involving top quarks into the product of the production cross section for on-shell top quarks and the top quark decay width. Most of the calculations mentioned above are performed for on-shell top quark pair production and we shall also follow this approach in this paper. Only most recently, the decay of the top quark has been included in NLO calculations [6, 7, 19, 20], leading to a similar theoretical accuracy of the NLO predictions.

At NNLO, intermediate results concerning the top-quark pair production cross section predictions have become available recently: Most notably, the inclusive total hadronic $t\bar{t}$ production cross section induced by the all-fermion partonic processes has been computed [21, 22]. NNLO calculations involving massive quarks require the same ingredients as their massless counterparts. Three classes of contributions enter: double real $d\sigma^{RR}$, mixed real-virtual, $d\sigma^{RV}$, and double virtual contributions $d\sigma^{VV}$.

Recent progresses has been accomplished concerning the two-loop contributions and many parts of the full top-pair NNLO matrix elements are known. Part of these two-loop virtual corrections are built with products of one-loop virtual amplitudes. Those corrections have been computed in [23–25]. Concerning the two-loop virtual corrections built with product of two-loop and tree-level amplitudes, the situation is different. The two-loop virtual corrections for the processes $q\bar{q} \rightarrow t\bar{t}$ and $gg \rightarrow t\bar{t}$ are not fully available at present. A purely numerical evaluation of the quark-initiated process [26] could be partly confirmed by analytical results [27–29], which were most recently extended also to the gluon-induced subprocess. The infrared structure of mixed-real virtual contributions has been studied in [30].

While infrared singularities from purely virtual corrections are obtained immediately after integration over the loop momenta, their extraction is more involved for real emission (or mixed real-virtual) contributions. There, the infrared singularities only become explicit after integrating the matrix elements over the phase space appropriate to the differential cross section under consideration. Since hadronic observables depend in general in a non trivial manner on the experimental criteria used to define them, they can only be calculated numerically. The computation of hadronic observables including higher order corrections therefore requires a systematic procedure to cancel infrared singularities among different partonic channels before any numerical computation of the observable can be performed.

Subtraction methods explicitly constructing infrared subtraction terms which coincide with the full matrix element in the unresolved limits and are sufficiently simple to be integrated analytically are well-known solutions to this problem. Starting from methods for subtraction at NLO [31–35], several NNLO subtraction methods have been proposed in the literature [30, 36–48], and have been worked out to a varying level of sophistication.

A subtraction method based on the transverse momentum structure of the final state has been proposed in [39]. Using this formalism, NNLO results were obtained for Higgs production and vector boson production [49–51]. This formalism has also recently been

applied to compute NNLO corrections to associated WH production [52] and photon pair production [53].

Other approaches to perform NNLO calculations of hadronic observables, concern the use of sector decomposition (possibly in combination with subtraction). The sector decomposition approach is a numerical method which avoids the need for analytical integration, and which has been developed for virtual [54–57] and real radiation [57–61] corrections to NNLO. It has been applied to several observables like Higgs production [62–65] or vector boson production [66].

It is the combination of subtraction with sector decomposition [30, 40, 41] which was also recently applied in the calculation of NNLO corrections to the all fermion-pair initiated processes in top quark pair production [21, 22] mentioned above.

Employing a subtraction method, the NNLO partonic cross section for a hadronic observable can be written as [67, 68]

$$\begin{aligned} d\hat{\sigma}_{NNLO} = & \int_{d\Phi_{m+2}} (d\hat{\sigma}_{NNLO}^{RR} - d\hat{\sigma}_{NNLO}^S) + \int_{d\Phi_{m+2}} d\hat{\sigma}_{NNLO}^S \\ & + \int_{d\Phi_{m+1}} (d\hat{\sigma}_{NNLO}^{RV} - d\hat{\sigma}_{NNLO}^{VS}) + \int_{d\Phi_{m+1}} d\hat{\sigma}_{NNLO}^{VS} + \int_{d\Phi_{m+1}} d\hat{\sigma}_{NNLO}^{MF,1} \\ & + \int_{d\Phi_m} d\hat{\sigma}_{NNLO}^{VV} + \int_{d\Phi_m} d\hat{\sigma}_{NNLO}^{MF,2}. \end{aligned} \quad (1.1)$$

In this equation, $d\hat{\sigma}_{NNLO}^S$ denotes the subtraction term for the $(m+2)$ -parton final state which behaves like the double real radiation contribution $d\hat{\sigma}_{NNLO}^{RR}$ in all singular limits. Likewise, $d\hat{\sigma}_{NNLO}^{VS}$ is the one-loop virtual subtraction term coinciding with the one-loop $(m+1)$ -final state $d\hat{\sigma}_{NNLO}^{RV}$ in all singular limits. The two-loop correction to the $(m+2)$ -parton final state is denoted by $d\hat{\sigma}_{NNLO}^{VV}$. In addition, as there are partons in the initial state, two mass factorisation contributions, $d\hat{\sigma}_{NNLO}^{MF,1}$ and $d\hat{\sigma}_{NNLO}^{MF,2}$, for the $(m+1)$ - and m -particle final states respectively, need to be taken into account.

In this paper, in order to evaluate the double real contributions to the hadronic top quark pair production arising from the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$, we shall employ the antenna subtraction formalism generalised to hadronic observables involving massive fermions developed in [69].

Originally, the antenna subtraction method has been developed for the production of massless partons in electron-positron annihilation [67, 70, 71]. The method was then generalised in order to deal with coloured initial states and massless final state partons in [68, 72–77]. The framework for the construction of NNLO antenna subtraction terms for hadronic jet observables has been set up in [68, 75] in the context of a proof-of-principle implementation of the contribution of the purely gluonic contributions to di-jet production at hadron colliders.

The formalism has also been expanded to include massive fermions. In [78–80] it was extended at the NLO level and employed to compute the real NLO corrections to the $t\bar{t}$ and $t\bar{t} + jet$ hadronic production cross section. In [69] we extended the method to the NNLO level, and derived the double real emission corrections to $t\bar{t}$ hadronic production cross section coming from all partonic channels involving fermions only. The aim of this

paper is to show how, within the framework presented in [69], partonic processes involving gluons can also be evaluated.

For QCD observables involving massive final state particles, fewer subtraction terms are required since the real matrix elements develop singular behaviours in fewer regions of phase space than in the massless case. Indeed, for those observables, QCD radiation from massive particles can lead to soft divergencies but cannot lead to strict collinear divergencies, since the mass acts as an infrared regulator. As a consequence, less divergent contributions arise and the infrared structure of such processes is expected to be simpler. The kinematics is however more involved due to the finite value of the parton masses and the integration of the subtraction terms is more difficult. In addition, in calculations of such observables, logarithms involving the ratio of the scales present in a given reaction occur. These finite logarithms are related to the singular behaviour of the matrix elements in the massless limit [78, 80–82]. Depending on the kinematics of the reactions involved, these logarithms, although finite, may or may not be enhanced. For the kinematical situation under consideration in this paper, i.e. the production of a heavy fermion pair whose mass is of the the same order as as the partonic center-of-mass energy, we shall ignore these finite logarithms as also done in [21, 22], since they are not expected to be large.

The paper is organised as follows: In section 2 we briefly review the generalisation of the antenna subtraction method presented in [69] enabling the construction of subtraction terms for the double real radiation contributions to $t\bar{t}$ production in hadronic collisions. In section 3 we give a list of all genuine new NNLO massive four-parton tree-level antennae required in our calculation while in section 4 we list the behaviour of these antennae in their infrared limits. In section 5, we first present the double soft behaviour of the real matrix element associated to the process $gg \rightarrow Q\bar{Q}q\bar{q}$. This enables us to construct the subtraction term capturing the double unresolved features of the double real matrix elements explicitly. All subtraction terms required to capture the single and double unresolved behaviour of the double real matrix elements in leading and subleading colour will be given in that section too. In section 6, we test the validity of the subtraction terms. We check that the ratio between the real radiation matrix elements and the corresponding subtraction terms approaches unity in all single and double unresolved limits. Finally, section 7 contains our conclusions and an outline. Two appendices are also enclosed: Appendix A contains a list of all tree-level three-parton antennae together with their infrared limits, while the known four-parton antennae that are used in the construction of our subtraction terms are given in Appendix B.

2. Antenna subtraction for double real radiation with massive final state fermions

The extension of the antenna subtraction method for double real emission corrections to observables involving heavy fermions in the final state has been extensively discussed in [69]. In this paper we consider the double real emission contributions to the hadronic $t\bar{t}$ production stemming from the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$. The treatment of this

contribution does not require any further extensions of the method. We shall therefore give here only a brief summary of the method and refer the reader to [69] for more details.

Within the antenna subtraction formalism, which uses colour ordering properties of amplitudes in a crucial manner, the subtraction terms are constructed using the fundamental factorisation properties of QCD amplitudes and phase spaces in their collinear and soft limits. These terms rely on the following essential ingredients:

- (1) A set of antenna functions of various types, which capture all unresolved radiation emitted between two hard partons, the radiators. These can be either massless or massive and depending on where the two radiators are located, in the initial or in the final state, we distinguish three types of antennae: final-final, initial-final and initial-initial.
- (2) An exact momentum conserving and Lorentz invariant phase space factorisation based on $3 \rightarrow 2$ and $4 \rightarrow 2$ mappings between on-shell partons in all three configurations.
- (3) Phase-space mappings defining the momenta present in the factorised form of the matrix elements in terms of the original momenta present in the real radiation matrix elements for which the subtraction terms are constructed.

The subtraction terms are then built with products of antenna functions and reduced matrix elements squared in all three configurations: final-final, initial-final and initial-initial.

Compared to the massless antenna formalism, the presence of massive partons in the final states modifies the subtraction terms in a non-trivial way. Parton masses lead to modified kinematics and have to be taken into account for the phase space factorisations [69]. Furthermore, non-vanishing masses also modify the soft behaviour of the real radiation matrix elements as will be explained in section 5.

2.1 Double real radiation contributions to heavy quark pair production

The method developed in [69] enables us to describe the computation of the double real radiation corrections to the process

$$pp \rightarrow Q\bar{Q} + (m-2)\text{jets}. \quad (2.1)$$

To set the notation, the leading order (LO) partonic contribution to this process reads

$$\begin{aligned} d\hat{\sigma}_{LO}(p_1, p_2) &= \mathcal{N}_{LO} \sum_{m=2} d\Phi_m(p_Q, p_{\bar{Q}}, p_5, \dots, p_{m-2}; p_1, p_2) \\ &\times \frac{1}{S_{m-2}} |\mathcal{M}_{m+2}(p_Q, p_{\bar{Q}}, p_5, \dots, p_{m-2}; p_1, p_2)|^2 J_m^{(m)}(p_Q, p_{\bar{Q}}, p_5, \dots, p_{m-2}), \end{aligned} \quad (2.2)$$

where p_1 and p_2 are the momenta of the initial state partons, the massive partons Q and \bar{Q} have momenta p_Q and $p_{\bar{Q}}$, while the momenta of the remaining $(m-2)$ massless final state partons are labelled $p_5 \dots p_{m-2}$. S_{m-2} is a symmetry factor for identical massless

partons in the final state. $J_m^{(m)}(p_Q, p_{\bar{Q}}, p_5, \dots, p_{m-2})$ is the jet function. It ensures that out of $(m-2)$ massless partons and a pair of heavy quarks Q and \bar{Q} present in the final state at the partonic level, an observable with a pair of heavy quark jets in association with $(m-2)$ jets is built. At this order, each massless or massive parton forms a jet on its own. The normalization factor \mathcal{N}_{LO} includes all QCD-independent factors as well as the dependence on the renormalised QCD coupling constant α_s and is process dependent. \sum_{m-2} denotes the sum over all configurations with $(m-2)$ massless partons. $d\Phi_m$ is the phase space for an m -parton final state containing $(m-2)$ massless and two massive partons with total four-momentum $p_1^\mu + p_2^\mu$. In $d = 4 - 2\epsilon$ space-time dimensions, this phase space takes the form:

$$d\Phi_m(p_Q, p_{\bar{Q}}, p_5, \dots, p_{m-2}; p_1, p_2) = \frac{d^{d-1}p_Q}{2E_Q(2\pi)^{d-1}} \frac{d^{d-1}p_{\bar{Q}}}{2E_{\bar{Q}}(2\pi)^{d-1}} \\ \times \frac{d^{d-1}p_5}{2E_5(2\pi)^{d-1}} \dots \frac{d^{d-1}p_{m-2}}{2E_{m-2}(2\pi)^{d-1}} (2\pi)^d \delta^d(p_1 + p_2 - p_Q - p_{\bar{Q}} - p_5 - \dots p_{m-2}). \quad (2.3)$$

In eq.(2.2), $|\mathcal{M}_{m+2}|^2$ denotes a colour-ordered tree-level matrix element squared with m final state partons, out of which two are massive and two are initial state partons. These terms only account for the leading colour contributions to the squared matrix elements, since subleading colour contributions involve in general interferences between sub-amplitudes with different colour orderings. However, to keep the notation simpler we denote these interference terms also as $|\mathcal{M}_{m+2}|^2$.

The double real emission contributions to $pp \rightarrow Q\bar{Q} + (m-2)\text{jets}$ at the partonic level read

$$d\hat{\sigma}_{NNLO}^{RR}(p_1, p_2) = \mathcal{N}_{NNLO}^{RR} \sum_m d\Phi_{m+2}(p_Q, p_{\bar{Q}}, p_5, \dots, p_m; p_1, p_2) \\ \times \frac{1}{S_m} |\mathcal{M}_{m+4}(p_Q, p_{\bar{Q}}, p_5, \dots, p_m; p_1, p_2)|^2 J_m^{(m+2)}(p_Q, p_{\bar{Q}}, p_5, \dots, p_m). \quad (2.4)$$

$$\equiv \mathcal{N}_{NNLO}^{RR} \sum_m d\Phi_{m+2}(p_3, \dots, p_{m+4}; p_1, p_2) \\ \times \frac{1}{S_m} |\mathcal{M}_{m+4}(p_3, \dots, p_{m+4}; p_1, p_2)|^2 J_m^{(m+2)}(p_3, \dots, p_{m+4}), \quad (2.5)$$

where the last line is obtained by relabelling all final state partons. In this equation, the jet function $J_m^{(m+2)}$ ensures that out of m massless partons and a $Q\bar{Q}$ pair, an observable with a pair of heavy quark jets in addition to $(m-2)$ jets, is built. The next-to-next-to leading order normalisation factor \mathcal{N}_{NNLO}^{RR} includes all QCD-independent factors as well as the dependence on the renormalised QCD coupling constant α_s . It is related to the normalisation factor present at leading order, \mathcal{N}_{LO} , which depends on the specific process and parton channel under consideration. For the process under consideration in this paper, this relation will be specified in section 5.

This NNLO contribution to the hadronic $t\bar{t}$ production cross section develops singularities if one or two final state partons are unresolved (soft or collinear). Depending on the colour connection between these unresolved partons, the following configurations must be distinguished [67, 68]:

- (a) One unresolved parton but the experimental observable selects only m jets.
- (b) Two colour-connected unresolved partons (colour-connected).
- (c) Two unresolved partons that are not colour-connected but share a common radiator (almost colour-unconnected).
- (d) Two unresolved partons that are well separated from each other in the colour chain (colour-unconnected).
- (e) Compensation terms for the over-subtraction of large angle soft emission.

This separation among subtraction contributions according to colour connection is valid in all final-final, initial-final or initial-initial configurations and, in any of them, the subtraction formulae have a characteristic structure in terms of the required antenna functions. This antenna structure has been derived for processes involving only massless partons, for the final-final and initial-final cases in [67, 73] and [68] for the initial-initial case. Note that the presence of massive partons in the final state does not modify the general structure of the subtraction terms required to match the unresolved features given above as explained in [69].

For the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$ considered in this paper, the configuration (c), the almost colour-unconnected case, and the configuration (e), regarding the treatment of large angle soft radiation, do not occur. On one hand, the colour structure of the amplitude for this process does not allow configuration (c) and, on the other hand, the absence of final state gluons forbids configuration (e). In the following, we shall therefore restrict ourselves to describe how the configurations (a), (b) and (d) are dealt with here while leaving the discussion of configurations (c) and (e) to be treated elsewhere.

2.2 Single unresolved radiation subtraction: $d\hat{\sigma}_{NNLO}^{S,a}$

The general form of the subtraction terms for single unresolved radiation associated to $Q\bar{Q}$ production in association with $(m-2)$ - jets can be taken from the expressions derived in [69, 78] in all three final-final (f-f), initial-final (i-f) and initial-initial (i-i) configurations.

These types of subtraction terms, denoted by $d\hat{\sigma}_{NNLO}^{S,a}$, involve the phase space for the production of $(m+2)$ partons, $d\Phi_{m+2}$, with two of them being massive, the three-parton antenna functions appropriate to the given configuration X_{ijk}^0 (f-f), $X_{i,jk}^0$ (i-f) and $X_{ik,j}^0$ (i-i), the colour-ordered reduced $(m+3)$ -parton amplitude squared $|\mathcal{M}_{m+3}|^2$ (with one parton less than the original amplitude squared) and the jet function $J_m^{(m+1)}$. Depending on the configuration considered, the momenta present in the reduced matrix elements and in the jet function are defined with a particular $3 \rightarrow 2$ mapping. These mappings have been defined previously in [67–69, 72].

In all configurations (f-f, i-f and i-i), the products of three-parton antennae and reduced matrix elements present in the $d\hat{\sigma}_{NNLO}^{S,a}$ subtraction terms, coincide with the double real radiation matrix element squared given in (2.5), when parton j is unresolved. At NNLO, however, the jet function $J_m^{(m+1)}$ allows one of the $(m+1)$ final state momenta to be unresolved and the reduced matrix elements present in the subtraction terms can have a

further unresolved parton. In this limit, the subtraction term $d\hat{\sigma}_{NNLO}^{S,a}$ does not coincide with the double real emission contribution given in eq.(2.5) and, as explained in detail in [67,68], the resulting spurious double unresolved singularities must cancel against pieces of the subtraction terms coming from other configurations. In this paper, these other configurations will be either (b) or (d), associated to two unresolved particles which are colour-connected or colour-unconnected respectively.

To define the integrated forms of these subtraction terms $d\hat{\sigma}_{NNLO}^{S,a}$ involving only three-parton antennae, an appropriate phase space factorisation needs to be considered in all three configurations. This factorisation has been derived in the massless case in [72, 74] and in the massive case in [78, 80].

2.3 Colour-connected double unresolved radiation subtraction: $d\hat{\sigma}_{NNLO}^{S,b}$

When two unresolved partons j and k are adjacent and colour-connected to two hard radiators labelled i and l , we build our subtraction term starting from the four-particle tree-level antennae in the corresponding configurations X_{ijkl} (f-f), $X_{i,jkl}$ (i-f) and $X_{il,jk}$ (i-i). By construction, these subtraction terms contain all colour-connected double unresolved limits of the $(m+4)$ -parton matrix element associated with partons j and k unresolved between radiators i and l . However, the four-parton antennae can also be singular in single unresolved limits associated with j or k where they do not coincide with limits of the matrix elements. To ensure that these subtraction terms are only active in the double unresolved limits of the real radiation matrix elements, we subtract the appropriate single unresolved limits of the four-particle tree-level antennae using products of two tree-level three-particle antennae in the appropriate configurations.

More precisely, the complete $d\hat{\sigma}_{NNLO}^{S,b}$ subtraction terms involve: the phase space for the production of $(m+2)$ partons, $d\Phi_{m+2}$, with two of them being massive, the colour-ordered reduced $(m+2)$ -parton amplitude squared $|\mathcal{M}_{m+2}|^2$, (with two partons less than the original matrix element squared), the four-parton antennae and products of three-parton antennae appropriate to the given configuration. In addition, it also contains they jet function $J_m^{(m)}$. The explicit expressions for these subtraction terms in final-final, initial-final and initial-initial configurations read,

$$\begin{aligned} d\hat{\sigma}_{NNLO}^{S,b,(ff)} &= \mathcal{N}_{NNLO} \sum_{\text{perms}} d\Phi_{m+2}(p_3, \dots, p_{m+4}; p_1, p_2) \frac{1}{S_{m+2}} \\ &\times \sum_{jk} (X_{ijkl}^0 - X_{ijk}^0 X_{IKl}^0 - X_{jkl}^0 X_{iJL}^0) \\ &\times |\mathcal{M}_{m+2}(\dots, I, L, \dots)|^2 J_m^{(m)}(\dots, p_I, p_L, \dots), \end{aligned} \quad (2.6)$$

$$\begin{aligned} d\hat{\sigma}_{NNLO}^{S,b,(if)} &= \mathcal{N}_{NNLO} \sum_{\text{perms}} d\Phi_{m+2}(p_3, \dots, p_{m+4}; p_1, p_2) \frac{1}{S_{m+2}} \\ &\times \sum_{i=1,2} \sum_{jk} (X_{i,jkl}^0 - X_{i,jk}^0 X_{I,Kl}^0 - X_{jkl}^0 X_{i,JL}^0) \\ &\times |\mathcal{M}_{m+2}(\dots, \hat{I}, L, \dots)|^2 J_m^{(m)}(\dots, p_L, \dots), \end{aligned} \quad (2.7)$$

$$\begin{aligned}
d\hat{\sigma}_{NNLO}^{S,b,(ii)} &= \mathcal{N}_{NNLO} \sum_{\text{perms}} d\Phi_{m+2}(p_3, \dots, p_{m+4}; p_1, p_2) \frac{1}{S_{m+2}} \\
&\times \sum_{il=1,2} \sum_{jk} (X_{il,jk}^0 - X_{l,jk}^0 X_{iL,K}^0 - X_{i,kj}^0 X_{lL,J}^0) \\
&\times |\mathcal{M}_{m+2}(\dots, \hat{I}, \hat{L}, \dots)|^2 J_m^{(m)}(\tilde{p}_3, \dots, \tilde{p}_{m+4}), \tag{2.8}
\end{aligned}$$

where the sum runs over all colour-adjacent pairs j, k and implies the appropriate selection of hard momenta i, l which, as usual, have three possible assignments of radiators. In all cases the $(m+2)$ -parton matrix element is evaluated with new on-shell momenta given by a momentum mapping appropriate in each configuration. Those $4 \rightarrow 2$ mappings have been derived in [68, 72, 80] in the massless case. In [69] we checked that the same mappings can also be applied in the presence of massive partons.

The relevant genuine NNLO massless and massive four-parton antenna functions required to define our $d\hat{\sigma}^{S,b}$ subtraction terms for the process $gg \rightarrow Q\bar{Q}q\bar{q}$ will be given in section 3 while their single and double unresolved limits will be given in section 4.

Together with the mappings required to define the momenta in the reduced matrix elements, the subtraction terms also need a factorized form of the phase space. The $2 \rightarrow (m+2)$ parton phase space needs to be factorised into an antenna phase space involving four-partons, the unresolved partons and the radiators, times a reduced $2 \rightarrow m$ phase space. This factorisation is different depending on whether the hard radiators are in the initial or in the final state. For the cases involving only massless particles the phase space factorisations for final-final, initial-final and initial-initial configurations have been derived in [68, 72, 80]. The presence of massive hard radiators introduces slight differences with respect to the massless case in the factorisation formulae for final-final and initial-final configurations, but the formula in the initial-initial configuration remains unaltered as the radiators are massless. These factorisation formulae involving massive particles were derived in the final-final case in [79] and in the initial-final case in [69]. In this way, the integrated four-parton antennae are obtained, in all three configurations, as the integral over the appropriate unresolved part of the phase space of the unintegrated forms of the antenna functions.

2.4 Colour-unconnected unresolved radiation subtraction: $d\hat{\sigma}_{NNLO}^{S,d}$

In this configuration, the two unresolved partons are pair-wise colour-unconnected. The subtraction terms are made of two disjoint three-parton tree-level antennae of the types X_{ijk}^0 (f-f), $X_{i,jk}^0$ (i-f) and $X_{ik,j}^0$ (i-i), which have no common momenta. Together with those, the subtraction terms contain the colour-ordered reduced $(m+2)$ -parton amplitude squared $|\mathcal{M}_{m+2}|^2$ (with two partons less than the original matrix element squared) and the jet function $J_m^{(m)}$. The reduced matrix element squared and the jet function, both involve remapped momenta which arise from two independent $3 \rightarrow 2$ mappings appropriate to the given configuration. The explicit expressions for this subtraction term have been given in [69].

This term captures the double unresolved behaviour associated to two colour-unconnected unresolved partons in the real matrix element squared and compensates spurious double unresolved features arising in subtraction terms for configurations (a).

To obtain the integrated form of this counterterm, the $(m + 2)$ -parton phase space is factorised into an m -parton phase space multiplied by two independent phase space factors for each of the two antennae. The integrated form is thus simply the product of two integrated three-parton antennae.

3. Antenna functions

In this section, we present the new massive four-parton antenna functions which are needed to define the subtraction terms for the double real radiation corrections to hadronic heavy quark pair production due to the process $gg \rightarrow Q\bar{Q}q\bar{q}$ and presented in section 5.

Antenna functions are the key ingredients needed to build subtraction terms in the antenna subtraction method and can also be used as evolution kernels in parton showers [83–86]. In general, these functions are denoted with the character X . Each antenna is determined by its particle content (hard radiators and unresolved particles) as well as by the pair of hard particles that it collapses onto in its singular limits. Antennae that collapse onto a quark-antiquark pair are $X = A$ for $qg\bar{q}$ and $qgg\bar{q}$, $X = B$ for $q\bar{q}'q'\bar{q}$ and $X = C$ for $qq\bar{q}\bar{q}$. Antennae that collapse onto a (anti) quark and a gluon are $X = D$ for qgg and $qggg$, and $X = E$ for $qq'\bar{q}'$ and $qq'\bar{q}'g$. Finally gluon-gluon antennae are $X = F$ for ggg and $gggg$, $X = G$ for $gq\bar{q}$ and $gq\bar{q}g$, and $X = H$ for $g\bar{q}q'\bar{q}'$. All antenna functions are derived from physical colour-ordered matrix elements squared. Each type of antenna is obtained from a different physical process: quark-antiquark antennae are related to processes of the form $\gamma^* \rightarrow q\bar{q} + (\text{partons})$ [87], quark-gluon antennae, to $\tilde{\chi} \rightarrow \tilde{g} + (\text{partons})$ [70], and gluon-gluon antennae are derived from $H \rightarrow (\text{partons})$ [71].

The tree-level antenna functions are obtained, both in the massless or massive case, by normalising the colour-ordered three-(or four)-parton tree-level squared matrix elements to the squared matrix element for the basic two-parton process, omitting couplings and colour factors:

$$X_{ijk}^0 = S_{ijk,IK} \frac{|\mathcal{M}_{ijk}^0|^2}{|\mathcal{M}_{IK}^0|^2} \quad (3.1)$$

$$X_{ijkl}^0 = S_{ijkl,IL} \frac{|\mathcal{M}_{ijkl}^0|^2}{|\mathcal{M}_{IL}^0|^2}. \quad (3.2)$$

S denotes the symmetry factor associated with the antenna, which accounts both for potential identical particle symmetries and for the presence of more than one antenna in the basic two-parton process.

The three and four-parton initial-final and initial-initial antenna functions are, in principle, defined by crossing one or two massless partons from the final to the initial state in the corresponding final-final antennae. The three-parton initial-final and initial-initial

antennae are denoted respectively by $X_{i,jk}^0$ and $X_{ik,j}^0$ while the corresponding four-parton antennae will be denoted by $X_{i,jkl}^0$ and $X_{ik,jl}^0$ ¹.

While at NLO only three-parton tree-level antennae are used, at NNLO also four-parton tree-level and three-parton one-loop antenna functions need to be considered. The latter, however, are not needed for the double real radiation contributions, and will not be discussed here.

In contrast with the massless antenna functions, massive antennae have explicit mass terms and they can be of two different natures: flavour-conserving and flavour-violating. The latter are derived from partonic processes with a flavour-violating vertex connecting radiators of different flavours, one of them being massless and the other massive. Massive three-parton flavour violating antennae have been derived in [78], and one specific B-type four-parton antenna of this kind has been derived in [69]. It was needed for the construction of the subtraction terms for the purely fermionic processes contributing to the hadronic production of a $t\bar{t}$ pair at NNLO.

For the subtraction terms that will be presented in section 5 for the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$, the three-parton antennae that are needed are of A, D, E and F-type in different configurations with massive and/or massless partons. All these antennae have been computed and integrated in [67,72,78,80], with the exception of one flavour-violating A-type antenna. This antenna function, which involves a gluon in the initial state and massive quark and a massless antiquark in the final state will be presented together with its integrated form and infrared limits in the appendix A. The unintegrated form of all the other required three-parton antennae will also be presented there.

The genuine NNLO four-parton antennae that are needed for the partonic process under consideration are of B, E and G-type. The B and G-type antennae are known and will be given in the appendix B. The E type antennae are new and shall be derived below.

For the labelling of the partons in the antenna functions used throughout this paper we shall use the following conventions: Massless quarks will be indexed with q while massive ones with Q and their mass with m_Q . Partons crossed to the initial state are denoted with a hat. The first and the last particles in the argument of a given antenna are the hard radiators, while the partons placed in between the radiators are the unresolved particles. In order to make the mass-dependence in the expressions of the antenna functions explicit, we define our invariants as $s_{ij} = 2p_i \cdot p_j$. Finally, for conciseness, the $\mathcal{O}(\epsilon)$ pieces of the antenna functions will be omitted.

The B-type antenna required to define our subtraction terms is massive and is only needed in its final-final form. It is obtained from the ratio of the matrix elements squared for the physical processes $\gamma^* \rightarrow Q\bar{Q}q\bar{q}$ and $\gamma^* \rightarrow Q\bar{Q}$, and it is used to subtract the infrared singularities associated to the emission of an unresolved $q\bar{q}$ pair between the massive $Q\bar{Q}$ radiator pair. This final-final massive B-type antenna is known in unintegrated and integrated form [79], and will be given in the appendix for completeness. Its infrared limits will be recalled in section 4.

¹It is worth noting that, in the initial-final case this crossing may not be unambiguous and may result in the necessity of a further splitting of the antenna as noted in [72,78].

We also use a massless four-parton G-type antenna in its initial-initial form. The expression of this antenna is obtained by crossing two gluons in the corresponding final-final massless G-type antenna, which has been defined in [67] as the ratio of the processes $H \rightarrow g g q \bar{q}$ and $H \rightarrow g g$. This antenna is needed to subtract the infrared singularities associated to the emission of a massless $q \bar{q}$ pair radiated between the initial state gluons. Its unintegrated form will be given in the appendix while its infrared limits, which have not been documented so far, will be presented in section 4. Note as well, that the integrated form of this antenna has just been derived in [77].

The new massive four-parton E-type antenna functions are needed in their initial-final form. Those can be obtained by crossing the gluon in the corresponding final-final E-type antennae. Those are derived from the ratio of the processes $\tilde{\chi} \rightarrow \tilde{g} g q \bar{q}$ and $\tilde{\chi} \rightarrow \tilde{g} g$ with the massive gluino \tilde{g} playing the role of the massive (anti) quark of mass m_Q . The full amplitude for the process $\tilde{\chi} \rightarrow \tilde{g} g q \bar{q}$ contains leading and subleading colour pieces [71]. By squaring the leading colour piece, in which the $q \bar{q}$ is emitted between the gluino and the gluon in the colour chain, the E_4^0 antenna is obtained, while, squaring the subleading colour piece, in which the gluon is emitted between the $q \bar{q}$ pair, the \tilde{E}_4^0 antenna is obtained.

The infrared limits of these E-type antennae derived below will be given in section 4. Note also that, there is on-going work [88] towards the integration of these initial-final massive antennae.

To account for the infrared limits associated to the emission of an unresolved $q \bar{q}$ pair between a massive (anti) quark and an initial state gluon we use the following antenna function,

$$\begin{aligned}
E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) = & \frac{1}{(Q^2 + m_Q^2)^2} \left\{ \frac{s_{13}s_{14}}{s_{12}s_{234}} + \frac{s_{14}}{s_{12}s_{34}s_{24}} [-s_{13}s_{23} + s_{13}^2 + s_{14}^2] \right. \\
& + \frac{s_{14}}{s_{12}s_{34}s_{134}} [2s_{14}s_{23} + 2s_{14}s_{24} + s_{23}^2 - s_{24}^2] \\
& + \frac{s_{14}}{s_{12}s_{24}s_{234}} [-s_{13}s_{23} + s_{13}^2 + s_{14}^2] + \frac{s_{14}}{s_{12}s_{24}} [2s_{13} + 2s_{14} - s_{23}] \\
& + \frac{4s_{14}s_{12}^2s_{23}}{s_{34}^2s_{134}s_{234}} + \frac{s_{14}}{s_{34}^2s_{134}} \left[+4s_{12}s_{14} + 4s_{14}s_{23} + 4s_{14}s_{24} \right. \\
& \left. + 8s_{12}s_{24} + 4s_{12}^2 + 4s_{23}s_{24} + 4s_{24}^2 \right] \\
& - \frac{s_{14}}{s_{34}s_{24}s_{134}} [2s_{12}s_{14} + s_{14}s_{23} - 2s_{14}^2 - s_{12}s_{23} - s_{12}^2] \\
& + \frac{s_{14}}{s_{34}s_{134}^2} [-4s_{12}s_{23} - 4s_{12}s_{24} - 2s_{12}^2 - 4s_{23}s_{24} - 2s_{23}^2 - 2s_{24}^2] \\
& - \frac{s_{14}}{s_{234}} + \frac{s_{14}^2}{s_{34}^2s_{134}^2} [-4s_{12}s_{23} - 4s_{12}s_{24} - 2s_{12}^2 - 4s_{23}s_{24} - 2s_{23}^2 - 2s_{24}^2] \\
& - \frac{1}{s_{12}s_{34}s_{234}} [s_{13}s_{14}^2 + s_{13}^2s_{14} - s_{13}^2s_{23} + s_{13}^3 + s_{14}^2s_{23} + s_{14}^3] \\
& - \frac{1}{s_{12}s_{34}} \left[s_{13}s_{14} - 3s_{13}s_{23} - s_{13}s_{24} + 2s_{13}^2 + s_{14}s_{23} - s_{14}s_{24} \right. \\
& \left. + 4s_{14}^2 + s_{23}s_{24} + s_{23}^2 \right] - \frac{1}{s_{12}s_{134}} [s_{13}s_{24} - s_{14}s_{23}
\end{aligned}$$

$$\begin{aligned}
& -2s_{14}s_{24} - s_{23}s_{24} + s_{24}^2 \Big] - \frac{1}{s_{12}} [s_{13} + 2s_{14} - s_{23} - s_{24}] \\
& - \frac{s_{12}}{s_{34}s_{134}s_{234}} [2s_{14}s_{23} + 8s_{14}^2 - 2s_{12}s_{23} + 2s_{12}^2 + 2s_{23}^2] \\
& - \frac{s_{23}}{s_{34}^2s_{234}} [4s_{13}s_{14} - 8s_{12}s_{13} + 4s_{13}s_{23} + 4s_{13}^2 + 4s_{14}s_{23} + 4s_{12}s_{23} \\
& + 4s_{12}^2] + \frac{s_{23}^2}{s_{34}^2s_{234}^2} [-4s_{13}s_{14} + 4s_{12}s_{13} - 2s_{13}^2 + 4s_{12}s_{14} - 2s_{14}^2 - 2s_{12}^2] \\
& + \frac{1}{s_{34}^2} [4s_{13}s_{12} - 4s_{13}s_{23} + 4s_{13}s_{24} - 2s_{13}^2 - 4s_{14}s_{12} - 4s_{14}s_{23} \\
& - 4s_{14}s_{24} + 4s_{12}s_{23} - 4s_{12}s_{24} - 2s_{12}^2 - 2s_{24}^2] \\
& + \frac{1}{s_{34}s_{134}} [s_{12}s_{14} + 7s_{14}s_{23} - s_{14}s_{24} + 7s_{14}^2 + s_{12}s_{23} + 7s_{12}s_{24} \\
& + 6s_{12}^2 + 2s_{23}s_{24} + s_{23}^2 + 3s_{24}^2] + \frac{1}{s_{34}s_{234}} [-6s_{12}s_{13} - 2s_{13}s_{23} + 4s_{13}^2 \\
& + 4s_{14}s_{23} + 6s_{14}^2 + 5s_{12}^2] + \frac{1}{s_{34}} [3s_{13} + 2s_{14} - 6s_{12} - 6s_{23} - s_{24}] \\
& + \frac{s_{23}}{s_{24}s_{234}^2} [2s_{13}s_{14} - 2s_{12}s_{13} + s_{13}^2 - 2s_{12}s_{14} + s_{14}^2 + s_{12}^2] \\
& - \frac{1}{s_{24}s_{134}s_{234}} [s_{12}s_{13}s_{14} - s_{13}s_{14}s_{23} - s_{13}s_{14}^2 + s_{12}s_{13}s_{23} + 3s_{12}^2s_{14} \\
& - 3s_{12}s_{14}^2 + s_{14}^3 - s_{12}^3] - \frac{1}{s_{24}s_{134}} [s_{12}s_{13} - s_{12}s_{14} - s_{14}^2 + s_{12}^2] \\
& - \frac{1}{s_{24}s_{234}} [2s_{12}s_{13} - 2s_{13}s_{23} - s_{13}^2 - 3s_{12}s_{14} - s_{14}s_{23} + 4s_{14}^2 \\
& + s_{12}s_{23}] + \frac{1}{s_{24}} [2s_{13} - 2s_{14} + s_{34}] \\
& + \frac{1}{s_{134}^2} [-2s_{12}s_{23} - 2s_{12}s_{24} - s_{12}^2 - 2s_{23}s_{24} - s_{23}^2 - s_{24}^2] \\
& + \frac{1}{s_{134}s_{234}} [s_{13}s_{14} + 3s_{12}s_{13} - s_{13}^2 - 4s_{12}s_{14} - s_{14}^2 - 3s_{12}^2] \\
& + \frac{1}{s_{134}} [-3s_{13} + 6s_{14} + 3s_{23} - 2s_{24}] + 2 \\
& + m_Q m_\chi \left[-\frac{s_{12}}{s_{24}s_{134}} + \frac{3}{s_{34}} - \frac{s_{14}}{s_{34}s_{134}} + \frac{s_{14}s_{23}}{s_{24}s_{34}s_{134}} - \frac{2}{s_{134}} \right. \\
& - \frac{s_{23}}{s_{24}s_{345}} + \frac{2s_{23}}{s_{34}s_{345}} - \frac{1}{s_{345}} - \frac{4s_{14}}{s_{12}s_{24}} + \frac{1}{s_{12}s_{34}} [s_{24} - s_{13}] \\
& + \frac{s_{14}s_{23}}{s_{12}s_{24}s_{34}} - \frac{s_{24}}{s_{12}s_{134}} - \frac{s_{14}}{s_{12}s_{34}s_{134}} [s_{23} + 3s_{24}] \\
& - \frac{2s_{14}^2}{s_{12}s_{34}^2s_{134}} [s_{23} + s_{24}] + \frac{s_{13}}{s_{345}s_{12}} - \frac{s_{14}s_{23}}{s_{24}s_{345}s_{12}} \\
& - \frac{2s_{13}s_{23}}{s_{34}s_{345}s_{12}} + \frac{2s_{23}^2}{s_{34}^2s_{345}s_{12}} [s_{13} + s_{14}] \\
& \left. + \frac{2}{s_{12}s_{34}^2} [s_{13}s_{23} + s_{14}s_{23} - s_{13}s_{24} + s_{14}s_{24}] \right]
\end{aligned}$$

$$\begin{aligned}
& +m_Q^2 \left[\frac{s_{12}}{s_{24}s_{134}} - \frac{4}{s_{34}} + \frac{2}{s_{34}s_{134}} [2s_{12} - s_{14} + s_{23} + 3s_{24}] + \frac{2s_{23}}{s_{34}s_{345}} \right. \\
& - \frac{1}{s_{134}s_{345}} [s_{12} + s_{14} + s_{23} + s_{24}] + \frac{s_{23}}{s_{24}s_{134}s_{345}} [s_{12} - s_{14}] \\
& - \frac{4s_{23}}{s_{34}s_{134}s_{345}} [s_{23} + s_{14}] + \frac{1}{s_{345}} + \frac{1}{s_{34}s_{12}} [4s_{14} - 2s_{23} - 3s_{24}] \\
& - \frac{2}{s_{34}s_{134}^2} [s_{12}^2 + 2s_{12}s_{23} + 2s_{12}s_{24} + s_{23}^2 + s_{24}^2 + 2s_{23}s_{24}] \\
& + \frac{1}{s_{12}s_{134}} [2s_{23} - s_{24}] + \frac{1}{s_{12}s_{34}s_{134}} [2s_{23}^2 + 5s_{14}s_{23} + 2s_{24}^2 - s_{14}s_{24}] \\
& + \frac{2}{s_{12}s_{34}^2s_{134}} [s_{14}^2s_{23} + s_{14}^2s_{24}] - \frac{2}{s_{12}s_{34}s_{345}} [-2s_{23}^2 + s_{13}s_{23} - 2s_{14}s_{23}] \\
& - \frac{2s_{23}^2}{s_{12}s_{34}^2s_{345}} [s_{13} + s_{14}] - \frac{2}{s_{12}s_{34}^2} [s_{13}s_{23} + s_{14}s_{23} - s_{13}s_{24} + s_{14}s_{24}] \\
& + \frac{2}{s_{12}} - \frac{2}{s_{12}^2} [s_{13} + s_{14} - s_{23} - s_{24}] - \frac{2s_{23}}{s_{12}s_{345}} \\
& \left. - \frac{2}{s_{12}^2s_{34}} [s_{13}^2 - 2s_{13}s_{23} + s_{14}^2 + s_{23}^2 + s_{24}^2 - 2s_{14}s_{24}] \right] \\
& + m_Q^3 m_\chi \left[\frac{4}{s_{12}^2} - \frac{2}{s_{12}s_{34}s_{134}} [s_{23} + s_{24}] \right] + \frac{2m_Q^4}{s_{12}s_{34}s_{134}} [s_{23} + s_{24}] \Big\} \\
& + \mathcal{O}(\epsilon),
\end{aligned} \tag{3.3}$$

where $Q^2 = -(p_1 + p_3 + p_4 - p_2)^2$, $m_\chi = \sqrt{Q^2}$, $s_{134} = s_{13} + s_{14} + s_{34}$ and $s_{234} = s_{34} - s_{23} - s_{24}$.

In order to account for the triple collinear limits that involve a massless $q\bar{q}$ pair and an initial state gluon in those sub-leading colour amplitudes in which the gluon is placed between the quark and the antiquark in the gluon chain, we employ the following \tilde{E}_4^0 antenna

$$\begin{aligned}
\tilde{E}_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) = & \frac{1}{(Q^2 + m_Q^2)^2} \left\{ \frac{1}{s_{23}s_{24}} \left[-2s_{12}s_{13} - 2s_{12}s_{14} - 2s_{12}s_{34} + 2s_{12}^2 \right. \right. \\
& + 2s_{13}s_{34} + 2s_{13}^2 + 2s_{14}s_{34} + 2s_{14}^2 \Big] + \frac{s_{23}}{s_{234}^2s_{24}} \left[-2s_{12}s_{13} - 2s_{12}s_{14} \right. \\
& + s_{12}^2 + 2s_{13}s_{14} + s_{13}^2 + s_{14}^2 \Big] + \frac{s_{24}}{s_{23}s_{234}^2} \left[-2s_{12}s_{13} - 2s_{12}s_{14} + s_{12}^2 \right. \\
& + 2s_{13}s_{14} + s_{13}^2 + s_{14}^2 \Big] - \frac{1}{s_{234}s_{24}} \left[4s_{12}s_{13} + 2s_{12}s_{14} + s_{12}s_{23} - 2s_{12}^2 \right. \\
& - 2s_{13}s_{14} - s_{13}s_{23} - 2s_{13}^2 - s_{14}s_{23} \Big] - \frac{1}{s_{23}s_{234}} \left[2s_{12}s_{13} + 4s_{12}s_{14} \right. \\
& + s_{12}s_{24} - 2s_{12}^2 - 2s_{13}s_{14} - s_{13}s_{24} - s_{14}s_{24} - 2s_{14}^2 \Big] \\
& \left. - m_Q m_\chi \left[\frac{2s_{24}}{s_{23}s_{234}} + \frac{4s_{234}}{s_{23}s_{24}} + \frac{2s_{23}}{s_{234}s_{24}} + \frac{4}{s_{23}} + \frac{4}{s_{24}} \right] \right\} + \mathcal{O}(\epsilon),
\end{aligned} \tag{3.4}$$

with Q^2 , m_χ , s_{134} and s_{234} given as above. Both E-type antennae are normalised to the tree-level two-parton matrix element

$$|\mathcal{M}_2^0(\tilde{\chi}g \rightarrow \tilde{g})|^2 = 4(1 - \epsilon) [Q^2 + m_Q^2]^2. \quad (3.5)$$

4. Infrared limits

The factorisation properties of QCD tree-level squared amplitudes have been extensively studied in [89–92]. While at NLO only single soft and collinear singularities may arise, at NNLO there are several double unresolved regions of phase space in which the amplitudes can develop singularities when integrated over. In the first part of this section, we shall list all single and double unresolved factors arising in the unresolved limits of the antennae needed to construct our subtraction term presented in section 5, while in the second part we shall list the infrared limits of all the required four-parton antennae.

4.1 Single unresolved factors

In their infrared limits, colour-ordered matrix elements factorise into universal unresolved factors and reduced matrix elements. Depending on whether the partons involved in each limit are massless or massive, and depending also on whether one or two partons become unresolved, the universal factors will be different. We shall start by presenting the single unresolved massless and massive factors.

Retaining only those infrared limits which lead to poles in ϵ when integrated over the phase space, the only single unresolved limits that the colour-ordered amplitudes for $gg \rightarrow Q\bar{Q}q\bar{q}$ have, are collinear limits involving the massless final state $q\bar{q}$ pair, and initial-final collinear limits involving one of the initial state gluons and the massless final state quark or antiquark. In the latter limit the collinear partons cluster into an initial state (anti) quark, while in the former, the collinear quark-antiquark pair clusters to form a gluon in the final state. As a consequence, the reduced colour-ordered matrix element associated with this limit involves a final state gluon. Thus, although only quarks are involved in the final state of the process that we are presently considering, unresolved gluon limits have to be considered as well.

When two massless partons become collinear, a colour-ordered sub-amplitude factorises into a reduced matrix element times a specific Altarelli-Parisi splitting function corresponding to the particular parton-parton splitting. These functions depend on z , the momentum fraction carried by the unresolved parton. Depending whether the unresolved parton is collinear to an initial or to a final state parton, the definition of z is different. For two final state particles i and j of momenta p_i and p_j becoming collinear, we have, in the limit,

$$p_i \rightarrow zp_{ij}, \quad p_j \rightarrow (1-z)p_{ij}, \quad s_{ik} \rightarrow zs_{ijk}, \quad s_{jk} \rightarrow (1-z)s_{ijk}, \quad (4.1)$$

whereas for a final state particle j of momentum p_j becoming collinear with an initial state parton i of momentum p_i we have

$$p_j \rightarrow zp_i, \quad p_{ij} \rightarrow (1-z)p_i, \quad s_{ik} \rightarrow \frac{s_{ijk}}{1-z}, \quad s_{jk} \rightarrow \frac{zs_{ijk}}{1-z}. \quad (4.2)$$

The splitting functions denoted by $P_{ij \rightarrow (ij)}(z)$ corresponding to the collinear limit of two final state partons i and j , given in the conventional dimensional regularisation scheme with all particles treated in $d = 4 - 2\epsilon$ dimensions, are [93]:

$$P_{qg \rightarrow Q}(z) = \frac{1 + (1 - z)^2 - \epsilon z^2}{z} \quad (4.3)$$

$$P_{q\bar{q} \rightarrow G}(z) = \frac{z^2 + (1 - z)^2 - \epsilon}{1 - \epsilon} \quad (4.4)$$

$$P_{gg \rightarrow G}(z) = 2 \left[\frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right]. \quad (4.5)$$

When the collinearity arises between an initial (i) and a final state parton (j), the splitting functions denoted by $P_{ij \leftarrow (ij)}(z)$ are given by [93]:

$$P_{gq \leftarrow Q}(z) = \frac{1 + z^2 - \epsilon(1 - z)^2}{(1 - \epsilon)(1 - z)^2} = \frac{1}{1 - z} \frac{1}{1 - \epsilon} P_{qg \rightarrow Q}(1 - z) \quad (4.6)$$

$$P_{qg \leftarrow Q}(z) = \frac{1 + (1 - z)^2 - \epsilon z^2}{z(1 - z)} = \frac{1}{1 - z} P_{qg \rightarrow Q}(z) \quad (4.7)$$

$$P_{q\bar{q} \leftarrow G}(z) = \frac{z^2 + (1 - z)^2 - \epsilon}{1 - z} = \frac{1 - \epsilon}{1 - z} P_{q\bar{q} \rightarrow G}(z) \quad (4.8)$$

$$P_{gg \leftarrow G}(z) = \frac{2(1 - z + z^2)^2}{z(1 - z)^2} = \frac{1}{1 - z} P_{gg \rightarrow G}(z). \quad (4.9)$$

The additional factors $(1 - \epsilon)$ and $1/(1 - \epsilon)$ account for the different number of polarizations of quark and gluons in the cases in which the particle entering the hard processes changes its type.

In all the splitting functions defined above, the label q can stand for a massless quark or an antiquark since charge conjugation implies that $P_{qg \rightarrow Q} = P_{\bar{q}g \rightarrow \bar{Q}}$ and $P_{qg \leftarrow Q} = P_{\bar{q}g \leftarrow \bar{Q}}$. The labels Q and G denote the parent parton of the two collinear partons, which is massless.

When a gluon j emitted between the hard radiators i and k becomes soft, the eikonal factor that factorises off the colour-ordered squared matrix element is

$$S_{ijk}(m_i, m_k) = \frac{2s_{ik}}{s_{ij}s_{jk}} - \frac{2m_i^2}{s_{ij}^2} - \frac{2m_k^2}{s_{jk}^2}, \quad (4.10)$$

where m_i and m_k are the masses of the radiators. This limit is obtained by letting $p_j \rightarrow \lambda p_j$ with $\lambda \rightarrow 0$ or, alternatively, by keeping only terms with any product of the inverse of the two invariants s_{ij} and s_{jk} in the colour-ordered matrix element squared. By letting $m_i, m_k \rightarrow 0$ the usual expression for the massless soft eikonal factor is obtained.

4.2 Double unresolved factors

In general, when two particles are unresolved in a tree-level process, a variety of different configurations can arise:

- (1) two soft particles,
- (2) two pairs of collinear particles,
- (3) three collinear particles,
- (4) one soft and two collinear.

As we saw in section 2, the resulting double unresolved configurations of the colour-ordered matrix element squared need to be separated into three categories depending on the colour connections of the unresolved partons and the hard radiators associated with them. In the following, we shall describe only the double unresolved factors encountered in the context of this paper, which correspond to the configurations (1), (2) and (3) listed above, but not to (4), since the process $gg \rightarrow Q\bar{Q}q\bar{q}$ does not have any soft-and-collinear limits.

4.2.1 Colour-unconnected pairs of unresolved particles

For unresolved particles that are disjoint in the colour chain, which arise in the item (d) as presented in section 2, the colour-ordered matrix elements squared factorise into the product of two disjoint single unresolved factors multiplied by a reduced matrix element with two partons less than the original colour-ordered matrix element squared.

For example, in the case of two pairs of colour-connected particles (a, b) and (c, d) which are all located in the final state, the colour-ordered matrix elements given by $|\mathcal{M}_n^0(\dots, a, b, \dots, c, d, \dots)|^2$ undergo the following factorisation in the double collinear limit:

$$|\mathcal{M}_n^0(\dots, a, b, \dots, c, d, \dots)|^2 \rightarrow P_{ab \rightarrow P}(z_1, s_{ab}) P_{cd \rightarrow Q}(z_2, s_{cd}) |\mathcal{M}_{n-2}^0(\dots, P, \dots, Q, \dots)|^2, \quad (4.11)$$

where partons a and b form P , while c and d cluster to form Q , so that P and Q are themselves colour-unconnected. The collinear splitting functions appearing in this equation are the (colourless) Altarelli-Parisi splitting functions given by eqs.(4.3-4.9).

In the process that we are considering in this paper, the only double unresolved colour-unconnected limits that can occur are double collinear (anti) quark-gluon limits, with each single collinear pair given by a final state (anti) quark and an initial state gluon. The unresolved factor associated to this limit is a product of two splitting functions of the type given in eqs.(4.3-4.9). Note that none of the four-parton antennae required for our subtraction term captures these double collinear singularities. These are accounted for by subtraction terms involving the product of two three-parton antennae.

4.2.2 Colour-connected pairs of unresolved particles

The colour-connected double unresolved limits the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$ develops are:

- A) Triple collinear limits involving an initial state gluon and a massless final state quark-antiquark pair.

- B) Double soft limits of a massless final state $q\bar{q}$ pair emitted between massive or massless radiators.

As mentioned above, these colour-connected double unresolved limits are captured by four-parton antenna functions and arise in subtraction terms $d\sigma_{NNLO}^{S,b}$ as presented in section 2. In the following, we shall give the massive and massless double unresolved factors associated with these two types of limits.

A) Massless triple collinear factors

In those regions of phase space where three colour-connected massless partons (a, b, c) become collinear, a generic colour-ordered amplitude squared denoted by $|\mathcal{M}_n^0(\dots, a, b, c, \dots)|^2$ factorises as:

$$|\mathcal{M}_n^0(\dots, a, b, c, \dots)|^2 \rightarrow P_{abc \rightarrow P} |\mathcal{M}_{n-2}^0(\dots, P, \dots)|^2. \quad (4.12)$$

where the three colour-connected final state particles (a, b, c) cluster to form a single parent particle P . The triple collinear splitting function for partons a, b and c clustering to form the parent parton P is generically denoted by,

$$P_{abc \rightarrow P}(w, x, y, s_{ab}, s_{ac}, s_{bc}, s_{abc}), \quad (4.13)$$

where w, x and y are the momentum fractions of the clustered partons,

$$p_a = wp_P, \quad p_b = xp_P, \quad p_c = yp_P, \quad \text{with } w + x + y = 1. \quad (4.14)$$

In addition to its dependence on the momentum fractions carried by the clustering partons, the splitting function also depends on the invariant masses of parton-parton pairs and the invariant mass of the whole cluster. The explicit forms of the triple collinear splitting functions $P_{abc \rightarrow P}$ are obtained by retaining terms in the colour-ordered matrix element squared that possess two of the ‘small’ denominators s_{ab}, s_{ac}, s_{bc} and s_{abc} .

The triple collinear limits which have to be considered here are those involving the massless final state quark-antiquark pair and one of the initial state gluons. The splitting functions for these types of (initial-final) triple collinear limits can be obtained from the analogous limit where the three collinear particles are in the final state [89, 92].

The clustering of a gluon with a quark-antiquark pair into a parent gluon has two distinct functions depending the colour connection of the collinear particles. In leading colour contributions where the gluon is emitted “outside” the quark-antiquark pair, the following non-abelian splitting function is obtained

$$\begin{aligned} P_{g\bar{q}q \rightarrow G}(w, x, y, s_{g\bar{q}}, s_{\bar{q}q}, s_{g\bar{q}q}) = & \\ & - \frac{1}{s_{g\bar{q}q}^2} \left(4 \frac{s_{g\bar{q}}}{s_{\bar{q}q}} + (1 - \epsilon) \frac{s_{\bar{q}q}}{s_{g\bar{q}}} + (3 - \epsilon) \right) - \frac{2(x s_{g\bar{q}q} - (1 - w) s_{g\bar{q}})^2}{s_{\bar{q}q}^2 s_{g\bar{q}q}^2 (1 - w)^2} \\ & + \frac{1}{s_{g\bar{q}} s_{g\bar{q}q}} \left(\frac{(1 - y)}{w(1 - w)} - y - 2w - \epsilon - \frac{2x(1 - y)(y - w)}{(1 - \epsilon)w(1 - w)} \right) \\ & + \frac{1}{s_{g\bar{q}} s_{\bar{q}q}} \left(\frac{x((1 - w)^3 - w^3)}{w(1 - w)} - \frac{2x^2(1 - yw - (1 - y)(1 - w))}{(1 - \epsilon)w(1 - w)} \right) \end{aligned}$$

$$+ \frac{1}{s_{\bar{q}q}s_{g\bar{q}q}} \left(\frac{(1+w^3+4xw)}{w(1-w)} + \frac{2x(w(x-y)-y(1+w))}{(1-\epsilon)w(1-w)} \right). \quad (4.15)$$

In those sub-leading colour pieces where the gluon is emitted “between” the quark-antiquark pair in the colour chain, one obtains a QED-like splitting function

$$\begin{aligned} \tilde{P}_{q\bar{q} \rightarrow G}(w, x, y, s_{qg}, s_{g\bar{q}}, s_{\bar{q}q}, s_{qg\bar{q}}) = \\ - \frac{1}{s_{q\bar{q}}^2} \left((1-\epsilon) \frac{s_{q\bar{q}}}{s_{qg}} + 1 \right) + \frac{1}{s_{g\bar{q}}s_{qg}} \left((1+x^2) - \frac{x+2wy}{1-\epsilon} \right) \\ - \frac{1}{s_{qg}s_{qg\bar{q}}} \left(1+2x+\epsilon - \frac{2(1-y)}{(1-\epsilon)} \right) + (s_{qg} \leftrightarrow s_{g\bar{q}}, w \leftrightarrow y). \end{aligned} \quad (4.16)$$

The triple collinear splitting functions given in eqs.(4.15) and (4.16) correspond to configurations in which all three collinear particles are outgoing. As mentioned above, in the present calculation we do not deal with these limits (indeed, there are not enough massless particles in the final state to have final-final triple collinear limits). Instead, we deal with triple collinear limits where one of the partons (a gluon) is in the initial state. In this case, initial-final triple collinear splitting functions arise. They are related to their final-final counterparts as follows [92]

$$\begin{aligned} P_{g\bar{q}q \leftarrow G}(z_3, z_2, z_1, s_{\hat{g}\bar{q}}, s_{q\bar{q}}, s_{\hat{g}q}) = \\ P_{g\bar{q}q \rightarrow G}(1/z_3, -z_2/z_3, -z_1/z_3, -s_{g\bar{q}}, s_{q\bar{q}}, s_{q\bar{q}} - s_{g\bar{q}} - s_{qg}) \\ \tilde{P}_{q\bar{q} \leftarrow G}(z_1, z_3, z_2, s_{q\hat{g}}, s_{q\bar{q}}, s_{q\hat{g}\bar{q}}) = \\ \tilde{P}_{q\bar{q} \rightarrow G}(-z_1/z_3, 1/z_3, -z_2/z_3, -s_{qg}, -s_{g\bar{q}}, s_{q\bar{q}}, s_{q\bar{q}} - s_{g\bar{q}} - s_{qg}), \end{aligned} \quad (4.17)$$

where z_1 and z_2 are the momentum fractions final state quark and antiquark respectively, and $z_3 = 1 - z_1 - z_2$.

B) Massive and massless double soft factors

When a massless quark-antiquark pair becomes soft between two hard radiators, sub-amplitudes squared factorise into a double soft factor and a reduced matrix element squared with the quark-antiquark pair removed from it. As it was the case for the single soft gluon eikonal factor, this double soft factor, needed for a soft $q\bar{q}$ pair, also depends on the masses of the hard radiators if these are massive.

When the quark-antiquark pair (c, d) becomes soft between the hard radiators (a, b) of masses m_a and m_b , the massive double soft factor is given by,

$$\begin{aligned} \mathcal{S}_{acdb}(m_a, m_b) = \frac{2(s_{ab}s_{cd} - s_{ac}s_{bd} - s_{bc}s_{ad})}{s_{cd}^2(s_{ac} + s_{ad})(s_{bc} + s_{bd})} + \frac{2}{s_{cd}^2} \left[\frac{s_{ac}s_{ad}}{(s_{ac} + s_{ad})^2} + \frac{s_{bc}s_{bd}}{(s_{bc} + s_{bd})^2} \right] \\ - \frac{2m_a^2}{s_{cd}(s_{ac} + s_{ad})^2} - \frac{2m_b^2}{s_{cd}(s_{bc} + s_{bd})^2}. \end{aligned} \quad (4.18)$$

This factor is obtained by setting $p_c \rightarrow \lambda p_c$, $p_d \rightarrow \lambda p_d$ with $\lambda \rightarrow 0$ in the matrix elements squared and was derived in [79] as the soft $q\bar{q}$ limit of the massive final-final antenna $B_4^0(Q, \bar{q}, q, \bar{Q})$ and in [69] using current algebra. Note also that, the corresponding massless factor [67, 89] can be obtained from this massive one by setting the masses to zero.

4.3 Single and double unresolved limits of four-parton antennae

In their infrared limits, the four-parton antennae of B, E and G types required to compute the double real contributions to heavy quark pair production in hadronic collisions due to the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$, and defined in section 3, yield the universal single and double unresolved factors defined above. Additionally, some of these four-parton antennae yield angular correlation terms in single collinear limits. These angular terms arise when a gluon splits into a quark-antiquark pair or into two gluons, and the way in which they are dealt with in the antenna subtraction method has been explained in detail in [68,69,94,95] and will not be recalled here.

In addition to the conventions already defined in section 3 for the labelling of the partons present in antenna function, we shall denote with $(ij)_\alpha$ the momentum of the parent partons in collinear limits. Thus, partons labelled with $(ij)_\alpha$ will have its momentum $p_i + p_j$ in final-final collinear limits, whereas its momentum will be given by $p_i - p_j$ in the initial-final case.

Massive final-final B-type antennae

The massive final-final B-type antenna given in eq.(B.1) is only singular when the massless $q\bar{q}$ pair is soft or collinear. The behaviour of this antenna in each of these limits is given by

$$B_4^0(1_Q, 4_{\bar{q}}, 3_q, 2_{\bar{Q}}) \xrightarrow{3_q, 4_{\bar{q}} \rightarrow 0} \mathcal{S}_{1342}(m_Q, m_Q) \quad (4.19)$$

$$B_4^0(1_Q, 4_{\bar{q}}, 3_q, 2_{\bar{Q}}) \xrightarrow{3_q || 4_{\bar{q}}} \frac{1}{s_{34}} P_{q\bar{q} \rightarrow G}(z) A_3^0(1_Q, (34)_g, 2_{\bar{Q}}) + \text{ang.}, \quad (4.20)$$

where (ang.) indicates the presence of angular correlation terms.

Massless initial-initial G-type antennae

The massless initial-initial G-type antenna given in the appendix in eq.(B.3) has the following infrared limits

$$G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{3_q, 4_{\bar{q}} \rightarrow 0} \mathcal{S}_{1342}(0, 0) \quad (4.21)$$

$$G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{1}_g || 3_q || 4_{\bar{q}}} P_{g\bar{q}q \leftarrow G}(z_3, z_2, z_1, s_{13}, s_{34}, s_{134}) \quad (4.22)$$

$$G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{2}_g || 3_q || 4_{\bar{q}}} P_{g\bar{q}q \leftarrow G}(z_3, z_2, z_1, s_{24}, s_{34}, s_{234}) \quad (4.23)$$

$$G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{3_q || 4_{\bar{q}}} \frac{1}{s_{34}} P_{q\bar{q} \rightarrow G}(z) F_3^0(\hat{1}_g, (34)_g, \hat{2}_g) + \text{ang.} \quad (4.24)$$

$$G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{1}_g || 3_q} \frac{1}{s_{13}} P_{q\bar{q} \leftarrow G}(z) G_3^0(\hat{2}_g, 4_{\bar{q}}, \widehat{(13)}_{\bar{q}}) \quad (4.25)$$

$$G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{2}_g || 4_{\bar{q}}} \frac{1}{s_{24}} P_{q\bar{q} \leftarrow G}(z) G_3^0(\hat{1}_g, 3_q, \widehat{(24)}_q). \quad (4.26)$$

Massive initial-final E and \tilde{E} -type antennae

The initial-final E-type antenna given in eq.(3.3) has a soft $q\bar{q}$ limit, a triple collinear limit,

a final-final and an initial-final single collinear limits. The behaviour of this antenna in each of these limits is

$$E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{3_q, 4_{\bar{q}} \rightarrow 0} \mathcal{S}_{1342}(m_Q, 0) \quad (4.27)$$

$$E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{2}_g || 3_q || 4_{\bar{q}}} P_{g\bar{q}q \leftarrow G}(z_3, z_2, z_1, s_{24}, s_{34}, s_{234}) \quad (4.28)$$

$$E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{3_q || 4_{\bar{q}}} \frac{1}{s_{34}} P_{q\bar{q} \rightarrow G}(z) D_3^0(1_Q, (34)_g, \hat{2}_g) + \text{ang.} \quad (4.29)$$

$$E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{2}_g || 4_{\bar{q}}} \frac{1}{s_{24}} P_{q\bar{q} \leftarrow G}(z) E_3^0(1_Q, 3_q, \widehat{(24)}_{\bar{q}}). \quad (4.30)$$

The initial-final massive antenna $D_3^0(Q, g, \hat{g})$ is obtained by crossing to the initial state one of the gluons in its final-final counterpart. As it can be seen in eq.(4.29), it is the antenna onto which $E_4^0(Q, q, \bar{q}, \hat{g})$ collapses to in its single $q||\bar{q}$ collinear limit. It will be given in the appendix A.

The sub-leading colour antenna \tilde{E}_4^0 has a QED-like triple collinear and two initial-final single collinear limits:

$$\tilde{E}_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{2}_g || 3_q || 4_{\bar{q}}} \tilde{P}_{qg\bar{q} \leftarrow G}(z_1, z_3, z_2, s_{23}, s_{24}, s_{34}, s_{234}), \quad (4.31)$$

$$\tilde{E}_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{2}_g || 4_{\bar{q}}} \frac{1}{s_{24}} P_{q\bar{q} \leftarrow G}(z) E_3^0(1_Q, 3_q, \widehat{(24)}_q) \quad (4.32)$$

$$\tilde{E}_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{2}_g) \xrightarrow{\hat{2}_g || 3_{\bar{q}}} \frac{1}{s_{23}} P_{q\bar{q} \leftarrow G}(z) E_3^0(1_Q, 4_{\bar{q}}, \widehat{(23)}_{\bar{q}}). \quad (4.33)$$

5. Top quark pair production at the LHC

In this section we shall present the double real emission contributions to $t\bar{t}$ production at the LHC due to the process $gg \rightarrow Q\bar{Q}q\bar{q}$. Together with these, we shall give their corresponding antenna subtraction terms, which capture all single and double unresolved limits of the leading and subleading colour pieces of the real radiation matrix elements squared.

5.1 Conventions

To facilitate the reading of our expressions, we shall closely follow the notation in [69, 78] for matrix elements and subtractions terms. The main points of our conventions are the following: The matrix elements denoted with \mathcal{M} represent colour-ordered sub-amplitudes in which the coupling constants and colour factors are omitted. Furthermore, to explicitly visualise the colour connection between particles in these colour-ordered amplitudes, a double semicolon is used in the labeling of the partons present in a given matrix element. This double semicolon is used for separating chains of colour-connected partons. Partons within a pair of double semicolons belong to a same colour chain, and adjacent partons within a colour chain are colour-connected. An antiquark (or an initial state quark) at the end of a colour chain and a like flavour quark (or initial state antiquark) at the beginning of a different colour chain are also colour-connected since the two chains merge in the collinear limit where the $q\bar{q}$ clusters into a gluon. We also denote gluons which are photon-like and

only couple to quark lines with the index γ instead of g . In sub-amplitudes where all gluons are photon-like no semicolons are used, since the concept of colour connection is not meaningful. A hat over the label of a certain parton indicates that it is an initial state particle (for example, $\hat{1}_q$ is an initial state quark with momentum p_1).

Concerning the notation in the subtraction terms, the conventions for the reduced matrix elements are the same as those for the real radiation matrix elements discussed above. The remapped final-state momenta are denoted with tildes and the remapped momenta of initial state hard radiators are denoted by a bar and a hat, as used in other papers [68, 69, 75]. In the four-parton antenna functions the hard radiators are “on the edges” and the unresolved particles are “in the middle”.

5.2 Double real radiation contributions

For two incoming hadrons, H_1, H_2 , the hadronic heavy quark pair production cross section may be written as

$$d\sigma = \sum_{a,b} \int \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} f_{a/1}(\xi_1, \mu_F) f_{b/2}(\xi_2, \mu_F) d\hat{\sigma}_{ab}(\xi_1 H_1, \xi_2 H_2, \mu_F, \mu_R) . \quad (5.1)$$

ξ_1 and ξ_2 are the momentum fractions of the partons of species a and b in both incoming hadrons, f_i being the corresponding parton distribution functions. $d\hat{\sigma}_{ab}$ denotes the parton-level scattering cross section for incoming partons a and b which depends on the renormalisation and factorisation scales denoted by μ_R and μ_F respectively. The partonic cross section $d\hat{\sigma}_{ab}$ has a perturbative expansion in the strong coupling α_s which itself depends on the renormalisation scale μ_R .

Following the general factorisation formula given in eq.(5.1) (if we omit the renormalisation and factorisation scale dependences) the contribution of the partonic process $gg \rightarrow Q\bar{Q}$ to the leading order cross section for heavy $Q\bar{Q}$ production in hadronic collisions is

$$d\sigma_{gg \rightarrow Q\bar{Q}}^{LO} = \int \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} f_g(\xi_1) f_g(\xi_2) d\hat{\sigma}_{gg \rightarrow Q\bar{Q}}^{LO}. \quad (5.2)$$

The leading order partonic differential cross section, written in terms of colour-ordered matrix elements, is given by

$$d\hat{\sigma}_{gg \rightarrow Q\bar{Q}}^{LO} = \mathcal{N}_{LO} d\Phi_2(p_1, p_2; p_3, p_4) \left[N_c (|\mathcal{M}_4^0(1_Q, \hat{3}_g, \hat{4}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_4^0(1_Q, \hat{4}_g, \hat{3}_g, 2_{\bar{Q}})|^2) - \frac{1}{N_c} |\mathcal{M}_4^0(1_Q, \hat{3}_g, \hat{4}_g, 2_{\bar{Q}})|^2 \right] J_2^{(2)}(p_1, p_2). \quad (5.3)$$

In this equation, the normalisation factor \mathcal{N}_{LO} reads

$$\mathcal{N}_{LO} = \frac{1}{2s} \times \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2} \times (N_c^2 - 1) \times \frac{1}{4(N_c^2 - 1)^2}, \quad (5.4)$$

where s is the hadronic center of mass energy, $(N_c^2 - 1)$ comes from the colour sum, the factor $1/4(N_c^2 - 1)^2$ accounts for the averaging over the spin and colour of the incoming

two gluons, and $(1/2s)$ is the hadron-hadron flux factor. The coupling is defined as usual: $\alpha_s = g^2/4\pi$, $C(\epsilon) = (4\pi)^\epsilon e^{-\epsilon\gamma}/8\pi^2$, and $\bar{C}(\epsilon) = (4\pi)^\epsilon e^{-\epsilon\gamma}$. This way of expressing the coupling factors, with each power of α_s accompanied by a power of $\bar{C}(\epsilon)$, keeps the coupling dimensionless in dimensional regularisation. The phase space for the production of a $Q\bar{Q}$ pair of momenta p_1 and p_2 is given by $d\Phi_2(p_1, p_2; p_3, p_4)$. In it, p_3 and p_4 are the momenta of the initial state gluons. The jet function denoted by $J_2^{(2)}$ ensures that the top and the antitop are in two separate jets.

The colour-ordered amplitudes \mathcal{M}_4^0 are related to the full amplitude $M_4^0(1_Q, 2_{\bar{Q}}, \hat{3}_g, \hat{4}_g)$ through the colour decomposition

$$M_4^0(1_Q, 2_{\bar{Q}}, \hat{3}_g, \hat{4}_g) = g^2(\sqrt{2})^2 \left((T^{a_3} T^{a_4})_{i_1 i_2} \mathcal{M}_4^0(1_Q, \hat{3}_g, \hat{4}_g, 2_{\bar{Q}}) + (T^{a_4} T^{a_3})_{i_1 i_2} \mathcal{M}_4^0(1_Q, \hat{4}_g, \hat{3}_g, 2_{\bar{Q}}) \right), \quad (5.5)$$

and the colour-ordered amplitude with photon-like gluons is given by

$$M_4^0(1_Q, \hat{3}_\gamma, \hat{4}_\gamma, 2_{\bar{Q}}) = \mathcal{M}_4^0(1_Q, \hat{3}_g, \hat{4}_g, 2_{\bar{Q}}) + \mathcal{M}_4^0(1_Q, \hat{4}_g, \hat{3}_g, 2_{\bar{Q}}). \quad (5.6)$$

The contribution of the $gg \rightarrow Q\bar{Q}q\bar{q}$ partonic channel to the double real radiation cross section for heavy quark pair production in hadronic collisions is

$$d\sigma_{gg \rightarrow Q\bar{Q}q\bar{q}}^{RR} = \int \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} f_g(\xi_1) f_g(\xi_2) d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{RR} \quad (5.7)$$

with the partonic cross section given by

$$d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{RR} = \mathcal{N}_{NNLO}^{RR} d\Phi_4(p_1, p_2, p_3, p_4; p_5, p_6) |M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{5}_g, \hat{6}_g)|^2 J_2^{(4)}(p_1, p_2, p_3, p_4). \quad (5.8)$$

The full matrix element squared $|M_6^0(\dots)|^2$ is summed over spin and colour. The normalisation factor \mathcal{N}_{NNLO}^{RR} accounts for the spin and colour averaging, the flux factor and the sum over the possible flavours of the massless $q\bar{q}$ pair. The jet function $J_2^{(4)}$ corresponds to the selection criteria of a 2-jet event: out of four partons, from which two are a $Q\bar{Q}$ pair, an event with two jets is built. Each of these two jets has the heavy quark Q or the heavy antiquark \bar{Q} in it. The additional partons present in a jet are either theoretically unresolved (soft or collinear) or not “seen” by the experimental resolution criteria.

The full matrix element for the process $gg \rightarrow Q\bar{Q}q\bar{q}$ has the following colour decomposition [78]

$$M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{5}_g, \hat{6}_g) = g^4(\sqrt{2})^2 \sum_{(i,j) \in P(5,6)} \left[(T^{a_i} T^{a_j})_{i_1 i_4} \delta_{i_3, i_2} \mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}) + (T^{a_i})_{i_1 i_4} (T^{a_j})_{i_3 i_2} \mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}}) + \delta_{i_1, i_4} (T^{a_i} T^{a_j})_{i_3 i_2} \mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}) - \frac{1}{N_c} (T^{a_i} T^{a_j})_{i_1 i_2} \delta_{i_3, i_4} \mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}) \right]$$

$$\begin{aligned}
& -\frac{1}{N_c}(T^{a_i})_{i_1 i_2}(T^{a_j})_{i_3 i_4}\mathcal{M}_6^0(1_Q, \hat{i}_g, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}}) \\
& -\frac{1}{N_c}\delta_{i_1, i_2}(T^{a_i}T^{a_j})_{i_3 i_4}\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}) \Big]. \quad (5.9)
\end{aligned}$$

Squaring eq.(5.9), summing and averaging over spin, colour, and quark flavour, and plugging our result in eq.(5.8) allows us to write the partonic double real radiation cross section in the following form

$$\begin{aligned}
d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}} = & \mathcal{N}_{LO}N_F \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2} d\Phi_4(p_1, p_2, p_3, p_4; p_5, p_6) \sum_{(i,j) \in P(5,6)} \Big[\\
& N_c^2 \Big(|\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})|^2 + |\mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}})|^2 \\
& + |\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 \Big) \\
& + |\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})|^2 + |\mathcal{M}_6^0(1_Q, \hat{i}_g, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}})|^2 \\
& + |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}})|^2 - |\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})|^2 \\
& - |\mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}})|^2 - |\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 \\
& + 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})^\dagger) \\
& + 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{j}_g, \hat{i}_g, 2_{\bar{Q}})^\dagger) \\
& + \text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{j}_g, 4_{\bar{q}}; ; 3_q, \hat{i}_g, 2_{\bar{Q}})^\dagger) \\
& - \text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{j}_g, \hat{i}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})^\dagger) \\
& - \text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{j}_g, \hat{i}_g, 2_{\bar{Q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{i}_g, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{i}_g, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, , 2_{\bar{Q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, , 2_{\bar{Q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}})^\dagger) \\
& - 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, 4_{\bar{q}}; ; 3_q, \hat{j}_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, , 2_{\bar{Q}}; ; 3_q, \hat{j}_g, \hat{i}_g, 4_{\bar{q}})^\dagger) \\
& + \frac{1}{N_c^2} \Big(\frac{1}{2} |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{i}_\gamma, \hat{j}_\gamma)|^2 - |\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})|^2 \\
& - |\mathcal{M}_6^0(1_Q, \hat{i}_g, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}})|^2 - |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}})|^2 \\
& + \text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, 4_{\bar{q}})^\dagger) \\
& - \text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, \hat{j}_g, \hat{i}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})^\dagger) \\
& - \text{Re}(\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, \hat{i}_g, 4_{\bar{q}})^\dagger) \\
& + 2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, \hat{i}_g, \hat{j}_g, 4_{\bar{q}})^\dagger)
\end{aligned}$$

$$+2\text{Re}(\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}) \mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, \hat{i}_g, 4_{\bar{q}}^\dagger)) \Big] J_2^{(4)}(p_1, p_2, p_3, p_4). \quad (5.10)$$

As it can be seen in eq.(5.10), this cross section does not only contain colour-ordered matrix elements squared, but also interference terms of two different colour-ordered matrix elements.

5.3 Subtraction terms

In this subsection we present the subtraction terms which capture all unresolved behaviour present in the real contributions given above in eq.(5.10) and related to the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$. As explained in section 2, the full subtraction term receives in this case three different contributions

$$d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^S = d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,a} + d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,b} + d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,d}. \quad (5.11)$$

The $d\hat{\sigma}^{S,a}$ terms subtract the single unresolved limits of the six-parton real radiation matrix elements. They are constructed as products of three-parton tree-level antenna functions and five-parton reduced matrix elements with remapped momenta. The $d\hat{\sigma}^{S,b}$ pieces are genuine NNLO subtraction terms which account for those double unresolved limits of the real radiation matrix element that involve a pair of colour-connected partons; in this case these can be double soft and triple collinear limits. Finally, the $d\hat{\sigma}^{S,d}$ terms, constructed as a product of two three-parton tree-level antenna functions and a four-parton reduced matrix element, capture the double collinear behaviour of the real matrix element squared and compensate for the over-subtraction of colour-unconnected double unresolved limits introduced in $d\hat{\sigma}^{S,a}$. In the following, we shall outline in some detail the derivation of $d\hat{\sigma}^{S,b}$ for which we follow a different approach than the one usually employed in the construction of antenna subtraction terms. Our starting point is the factorisation of colour sub-amplitudes in their soft $q\bar{q}$ limits.

5.3.1 Double soft behaviour of amplitudes

As it was explained in [69] when a soft quark-antiquark pair is emitted between partons a and b in the colour chain, the colour-ordered amplitude denoted by $\mathcal{M}_n^0(\dots, a, c_{\bar{q}}; ; d_q, b, \dots)$ factorises as

$$\mathcal{M}_n^0(\dots, a, c_{\bar{q}}; ; d_q, b, \dots) \xrightarrow{p_c, p_d \rightarrow 0} [\bar{u}_{s_d}(p_d) \gamma_\mu v_{s_c}(p_c)] (J_a^\mu(p_c, p_d) - J_b^\mu(p_c, p_d)) \mathcal{M}_{n-2}^0(\dots, a, b, \dots), \quad (5.12)$$

where the soft currents are given by

$$J_i^\mu(p_j, p_k) = \frac{p_i^\mu}{s_{jk}(s_{ij} + s_{ik})}. \quad (5.13)$$

This factorisation can be applied to the first three lines of eq.(5.9). Squaring eq.(5.12) and summing over the spins of the soft quark and antiquark gives

$$|\mathcal{M}_n^0(\dots, a, c_{\bar{q}}; ; d_q, b, \dots)|^2 \xrightarrow{p_c, p_d \rightarrow 0} \mathcal{S}_{acdb}(m_a, m_b) |\mathcal{M}_{n-2}^0(\dots, a, b, \dots)|^2 \quad (5.14)$$

with the double soft factor $\mathcal{S}_{acdb}(m_a, m_b)$ given by

$$\mathcal{S}_{acdb}(m_a, m_b) = \text{tr}(\not{p}_d \gamma_\mu \not{p}_c \gamma_\nu) (J_a^\mu(p_c, p_d) - J_b^\mu(p_c, p_d)) (J_a^\nu(p_c, p_d) - J_b^\nu(p_c, p_d))^\dagger. \quad (5.15)$$

By explicitly evaluating the trace in eq.(5.15) and using the definition of the currents in eq.(5.13) the expression given in eq.(4.18) for the double soft factor is obtained.

When a $q\bar{q}$ pair becomes soft in a sub-amplitude whose colour factor contains δ_{i_d, i_c} , like $\mathcal{M}_6^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}})$ in the fourth line of eq.(5.9), the factorisation given in eq.(5.12) does not hold. This is due to the fact that the presence of the δ_{i_d, i_c} factor indicates that the gluon propagator from which the soft pair splits is photon-like. In these cases, factorisation at the amplitude level reads

$$\mathcal{M}_n^0(...., a; ; d_q, c_{\bar{q}}) \xrightarrow{p_c, p_d \rightarrow 0} [\bar{u}_{s_d}(p_d) \gamma_\mu v_{p_c}(p_c)] \left(\sum_{i \in \{q\}} J_i^\mu(p_c, p_d) - \sum_{j \in \{\bar{q}\}} J_j^\mu(p_c, p_d) \right) \mathcal{M}_{n-2}^0(...., a) \quad (5.16)$$

where $\{q\}$ is the set of all final state quarks and initial state antiquarks in the partonic process, and $\{\bar{q}\}$ is the set of all final state antiquarks and initial state quarks. In the process $gg \rightarrow Q\bar{Q}q\bar{q}$, however, since there is only one quark-antiquark pair in addition to the soft one, each sum in eq.(5.16) contains only one term. Also, the sub-amplitudes in the last two lines of eq.(5.9) are finite in the soft $q\bar{q}$ limit as the quark and the antiquark are not colour connected.

With these considerations, we can obtain the double soft behaviour of the full amplitude for the process $gg \rightarrow Q\bar{Q}q\bar{q}$ by readily applying eqs.(5.12,5.16) to eq.(5.9). It reads,

$$\begin{aligned} M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{5}_g, \hat{6}_g) &\xrightarrow{p_3, p_4 \rightarrow 0} g^4 (\sqrt{2})^2 [\bar{u}_{s_3}(p_3) \gamma_\mu v_{s_4}(p_4)] \\ &\times \sum_{(i,j) \in P(5,6)} \left[(T^{a_i} T^{a_j})_{i_1 i_4} \delta_{i_3, i_2} \left(J_j^\mu(p_4, p_3) - J_2^\mu(p_4, p_3) \right) \right. \\ &\quad + (T^{a_i})_{i_1 i_4} (T^{a_j})_{i_3 i_2} \left(J_i^\mu(p_4, p_3) - J_j^\mu(p_4, p_3) \right) \\ &\quad + \delta_{i_1, i_4} (T^{a_i} T^{a_j})_{i_3 i_2} \left(J_1^\mu(p_4, p_3) - J_i^\mu(p_4, p_3) \right) \\ &\quad \left. - \frac{1}{N_c} (T^{a_i} T^{a_j})_{i_1 i_2} \delta_{i_3, i_4} \left(J_1^\mu(p_4, p_3) - J_2^\mu(p_4, p_3) \right) \right] \mathcal{M}_4^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}}). \end{aligned} \quad (5.17)$$

We can now square eq.(5.17) and evaluate the Dirac traces as well as the colour traces to obtain the double soft limit of the full matrix element squared in terms of reduced colour-ordered matrix elements squared (with two partons less than the original amplitude) and combinations of double soft factors:

$$\begin{aligned} |M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{5}_g, \hat{6}_g)|^2 &\xrightarrow{p_3, p_4 \rightarrow 0} g^8 (N_c^2 - 1) \\ &\times \sum_{(i,j) \in P(5,6)} \left[N_c^2 \left(\mathcal{S}_{143i}(m_Q, 0) + \mathcal{S}_{j432}(0, m_Q) + \mathcal{S}_{i43j}(0, 0) \right) |\mathcal{M}_4^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 \right. \\ &\quad \left. - \mathcal{S}_{1432}(m_Q, m_Q) \left(|\mathcal{M}_4^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 - (1/2) |\mathcal{M}_4^0(1_Q, \hat{i}_\gamma, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \right) \right] \end{aligned}$$

$$\begin{aligned}
& - \left(\mathcal{S}_{143i}(m_Q, 0) + \mathcal{S}_{j432}(0, m_Q) \right) |\mathcal{M}_4^0(1_Q, \hat{i}_\gamma, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \\
& + \frac{1}{2N_c^2} \mathcal{S}_{1432}(m_Q, m_Q) |\mathcal{M}_4^0(1_Q, \hat{i}_\gamma, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \Big]. \tag{5.18}
\end{aligned}$$

In eq.(5.18), we see that in the soft factors the hard radiators are two particles taken from the list $\{1_Q, 2_{\bar{Q}}, \hat{i}_g, \hat{j}_g\}$, and depending on which of them are involved (massive or massless) the soft factors will contain two, one, or no mass terms.

5.4 The construction of $d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,b}$

We decompose the $d\hat{\sigma}^{S,b}$ subtraction term into a part which can be directly related to the double soft limit of the amplitude squared which we denote $d\hat{\sigma}^{S,b(1)}$ and another that accounts for all other double unresolved behaviour of the real matrix element for the process $gg \rightarrow Q\bar{Q}q\bar{q}$ which is not accounted for in $d\hat{\sigma}^{S,b(1)}$. We shall denote this latter part of the subtraction term as $d\hat{\sigma}^{S,b(2)}$ such that,

$$d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,b} = d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,b(1)} + d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,b(2)}. \tag{5.19}$$

From eqs.(4.19,4.21,4.27) we know that $\mathcal{S}_{1432}(m_Q, m_Q)$ is the double soft factor present in the double soft limit of $B_4^0(1_Q, 4_{\bar{q}}, 3_q, 2_{\bar{Q}})$, $\mathcal{S}_{143i}(m_Q, 0)$ appears in the double soft limit of $E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{i}_g)$, $\mathcal{S}_{243i}(m_Q, 0)$ appears in the double soft limit of $E_4^0(2_{\bar{Q}}, 4_{\bar{q}}, 3_q, \hat{i}_g)$, and finally $\mathcal{S}_{i43j}(0, 0)$ is the double soft limit of $G_4^0(\hat{i}_g, 3_q, 4_{\bar{q}}, \hat{j}_g)$.

The subtraction term $d\hat{\sigma}^{S,b(1)}$ is therefore obtained by replacing in eq.(5.18) the double soft factors with the corresponding four-parton antennae, remapping the momenta in the reduced matrix elements accordingly, and removing all spurious single collinear limits of the four-parton antennae with terms of the form $X_3^0 \cdot X_3^0$. It reads,

$$\begin{aligned}
d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,b(1)} &= \mathcal{N}_{LO} N_F \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2} d\Phi_4(p_1, p_2, p_3, p_4; p_5, p_6) \sum_{(i,j) \in P(5,6)} \left\{ \right. \\
& N_c^2 \left[\left(E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{i}_g) - G_3^0(\hat{i}_g, 3_q, 4_{\bar{q}}) D_3^0(1_Q, (\widetilde{34})_g, \hat{i}_g) \right. \right. \\
& \quad \left. \left. - A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) E_3^0((\widetilde{14})_Q, 3_q, \hat{i}_g) \right) |\mathcal{M}_4^0((\widetilde{134})_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 J_2^{(2)}(\widetilde{p_{134}}, p_2) \right. \\
& + \left(E_4^0(2_{\bar{Q}}, 4_{\bar{q}}, 3_q, \hat{i}_g) - G_3^0(\hat{i}_g, 3_q, 4_{\bar{q}}) D_3^0(2_{\bar{Q}}, (\widetilde{34})_g, \hat{i}_g) \right. \\
& \quad \left. \left. - A_3^0(2_{\bar{Q}}, \hat{i}_g, 3_q) E_3^0((\widetilde{23})_{\bar{Q}}, 4_{\bar{q}}, \hat{i}_g) \right) |\mathcal{M}_4^0(1_Q, \hat{j}_g, \hat{i}_g, (\widetilde{234})_{\bar{Q}})|^2 J_2^{(2)}(p_1, \widetilde{p_{234}}) \right. \\
& + \left(G_4^0(\hat{i}_g, 3_q, 4_{\bar{q}}, \hat{j}_g) - (1/2) G_3^0(\hat{i}_g, 3_q, 4_{\bar{q}}) F_3^0(\hat{i}_g, (\widetilde{34})_g, \hat{j}_g) \right. \\
& \quad \left. \left. - (1/2) G_3^0(\hat{j}_g, 3_q, 4_{\bar{q}}) F_3^0(\hat{i}_g, (\widetilde{34})_g, \hat{j}_g) - a_3^0(3_q, \hat{i}_g, 4_{\bar{q}}) G_3^0(\hat{j}_g, (\widetilde{34})_{\bar{q}}, \hat{i}_g) \right. \right. \\
& \quad \left. \left. - a_3^0(4_{\bar{q}}, \hat{j}_g, 3_q) G_3^0(\hat{i}_g, (\widetilde{34})_q, \hat{j}_g) \right) |\mathcal{M}_4^0(\widetilde{1}_Q, \hat{i}_g, \hat{j}_g, \widetilde{2}_{\bar{Q}})|^2 J_2^{(2)}(\widetilde{p_1}, \widetilde{p_2}) \right] \\
& - \left(E_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{i}_g) - G_3^0(\hat{i}_g, 3_q, 4_{\bar{q}}) D_3^0(1_Q, (\widetilde{34})_g, \hat{i}_g) \right. \\
& \quad \left. \left. - A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) E_3^0((\widetilde{14})_Q, 3_q, \hat{i}_g) \right) |\mathcal{M}_4^0((\widetilde{134})_Q, \hat{i}_\gamma, \hat{j}_\gamma, 2_{\bar{Q}})|^2 J_2^{(2)}(\widetilde{p_{134}}, p_2) \right\}
\end{aligned}$$

$$\begin{aligned}
& - \left(E_4^0(2_{\bar{Q}}, 4_{\bar{q}}, 3_q, \hat{i}_g) - G_3^0(\hat{i}_g, 3_q, 4_{\bar{q}}) D_3^0(2_{\bar{Q}}, (\widetilde{34})_g, \hat{i}_g) \right. \\
& \quad \left. - A_3^0(2_{\bar{Q}}, \hat{i}_g, 3_{\bar{q}}) E_3^0((\widetilde{23})_{\bar{Q}}, 4_{\bar{q}}, \hat{i}_{\bar{q}}) \right) |\mathcal{M}_4^0(1_Q, \hat{j}_\gamma, \hat{i}_\gamma, (\widetilde{234})_{\bar{Q}})|^2 J_2^{(2)}(p_1, \widetilde{p_{234}}) \\
& - \left(B_4^0(1_Q, 4_{\bar{q}}, 3_q, 2_{\bar{Q}}) - (1/2) E_3^0(1_Q, 3_q, 4_{\bar{q}}) A_3^0((\widetilde{14})_Q, (\widetilde{34})_g, 2_{\bar{Q}}) \right. \\
& \quad \left. - (1/2) E_3^0(2_Q, 3_q, 4_{\bar{q}}) A_3^0(1_Q, (\widetilde{34})_g, (\widetilde{24})_{\bar{Q}}) \right) \times \left(|\mathcal{M}_4^0((\widetilde{134})_Q, \hat{i}_g, \hat{j}_g, (\widetilde{234})_{\bar{Q}})|^2 \right. \\
& \quad \left. - (1/2) |\mathcal{M}_4^0((\widetilde{134})_Q, \hat{i}_\gamma, \hat{j}_\gamma, (\widetilde{234})_{\bar{Q}})|^2 \right) J_2^{(2)}(\widetilde{p_{134}}, \widetilde{p_{234}}) \\
& + \frac{1}{N_c^2} \left[\left(B_4^0(1_Q, 4_{\bar{q}}, 3_q, 2_{\bar{Q}}) - (1/2) E_3^0(1_Q, 3_q, 4_{\bar{q}}) A_3^0((\widetilde{14})_Q, (\widetilde{34})_g, 2_{\bar{Q}}) \right. \right. \\
& \quad \left. \left. - (1/2) E_3^0(2_Q, 3_q, 4_{\bar{q}}) A_3^0(1_Q, (\widetilde{34})_g, (\widetilde{24})_{\bar{Q}}) \right) \right. \\
& \quad \left. \times |\mathcal{M}_4^0((\widetilde{134})_Q, \hat{i}_\gamma, \hat{j}_\gamma, (\widetilde{234})_{\bar{Q}})|^2 J_2^{(2)}(\widetilde{p_{134}}, \widetilde{p_{234}}) \right] \Bigg\}. \tag{5.20}
\end{aligned}$$

As explained in section 4, the initial-final massive D_3^0 antenna appearing in this subtraction term, is the “full” antenna. It is this antenna onto which the E_4^0 antenna function collapses to in its single $q\bar{q}$ collinear limit.

In order to obtain the full $d\hat{\sigma}^{S,b}$ subtraction term, we have to supplement eq.(5.20) with additional terms that ensure the correct subtraction of all other colour-connected double unresolved limit of the real radiation matrix element squared. The only other unresolved limits of this type that the process $gg \rightarrow Q\bar{Q}q\bar{q}$ has are triple collinear limits $\hat{i}_g||3_q||4_q$. In these limits the full matrix element squared factorises as [89–92]

$$\begin{aligned}
|M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{5}_g, \hat{6}_g)|^2 \xrightarrow{\hat{i}_g||3_q||4_q} g^2 \left[N_c (P_{g_i\bar{q}_4q_3 \leftarrow G} + P_{\bar{q}_4q_3g_i \leftarrow G}) \right. \\
\left. - \frac{1}{N_c} P_{q_3g_i\bar{q}_4 \leftarrow G} \right] |M_4^0(1_Q, 2_{\bar{Q}}, (\widehat{34i})_g, \hat{j}_g)|^2. \tag{5.21}
\end{aligned}$$

Taking now the triple collinear limit of $d\hat{\sigma}^{S,b(1)}$ we obtain,

$$\begin{aligned}
d\hat{\sigma}_{g\bar{g} \rightarrow Q\bar{Q}q\bar{q}}^{S,b(1)} \xrightarrow{\hat{i}_g||3_q||4_q} \mathcal{N}_{LO} \mathcal{N}_F \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2} d\Phi_4(p_1, p_2, p_3, p_4; p_5, p_6) \\
\times N_c (P_{g_i\bar{q}_4q_3 \leftarrow G} + P_{\bar{q}_4q_3g_i \leftarrow G}) |M_4^0(1_Q, 2_{\bar{Q}}, (\widehat{34i})_g, \hat{j}_g)|^2 J_2^{(2)}(p_1, p_2). \tag{5.22}
\end{aligned}$$

where we have used eqs.(5.5,5.6) to relate the full four-parton reduced matrix element squared denoted by $|M_4^0(1_Q, 2_{\bar{Q}}, (\widehat{34i})_g, \hat{j}_g)|^2$ to the corresponding colour-ordered amplitudes. The triple collinear splitting functions $P_{ijk \leftarrow G}$ have been defined in section 4. For conciseness, in the above equations the arguments of these splitting functions have been omitted.

We see that the subtraction term $d\hat{\sigma}^{S,b(1)}$, originally derived in order to capture the soft $q\bar{q}$ limit, also subtracts the non-abelian piece of the triple collinear limits $\hat{i}_g||3_q||4_q$. To account for the QED-like triple collinear limits, proportional to $1/N_c$ in eq.(5.21), we use

initial-final \tilde{E}_4^0 massive four-parton antennae. The following subtraction term is obtained:

$$\begin{aligned}
d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,b(2)} = & \mathcal{N}_{LO} N_F \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2} d\Phi_4(p_1, p_2, p_3, p_4; p_5, p_6) \sum_{(i,j) \in P(5,6)} \left\{ \right. \\
& -\frac{1}{2} \left(\tilde{E}_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{i}_g) - A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) E_3^0((\widetilde{14})_Q, 3_q, \hat{i}_q) \right. \\
& \quad \left. - A_3^0(1_Q, \hat{i}_g, 3_q) E_3^0((\widetilde{13})_Q, 4_{\bar{q}}, \hat{i}_{\bar{q}}) \right) \times \left(|\mathcal{M}_4^0((\widetilde{134})_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 \right. \\
& \quad \left. + |\mathcal{M}_4^0((\widetilde{134})_Q, \hat{j}_g, \hat{i}_g, 2_{\bar{Q}})|^2 \right) J_2^{(2)}(\widetilde{p_{134}}, p_2) \\
& -\frac{1}{2} \left(\tilde{E}_4^0(2_{\bar{Q}}, 4_{\bar{q}}, 3_q, \hat{i}_g) - A_3^0(2_{\bar{Q}}, \hat{i}_g, 3_q) E_3^0((\widetilde{23})_{\bar{Q}}, 4_{\bar{q}}, \hat{i}_{\bar{q}}) \right. \\
& \quad \left. - A_3^0(2_{\bar{Q}}, \hat{i}_g, 4_{\bar{q}}) E_3^0((\widetilde{24})_Q, 3_q, \hat{i}_q) \right) \times \left(|\mathcal{M}_4^0(1_Q, \hat{i}_g, \hat{j}_g, (\widetilde{234})_{\bar{Q}})|^2 \right. \\
& \quad \left. + |\mathcal{M}_4^0(1_Q, \hat{j}_g, \hat{i}_g, (\widetilde{234})_{\bar{Q}})|^2 \right) J_2^{(2)}(p_1, \widetilde{p_{234}}) \\
& + \frac{1}{N_c^2} \left[\frac{1}{2} \left(\tilde{E}_4^0(1_Q, 3_q, 4_{\bar{q}}, \hat{i}_g) - A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) E_3^0((\widetilde{14})_Q, 3_q, \hat{i}_q) \right. \right. \\
& \quad \left. \left. - A_3^0(1_Q, \hat{i}_g, 3_q) E_3^0((\widetilde{13})_Q, 4_{\bar{q}}, \hat{i}_{\bar{q}}) \right) |\mathcal{M}_4^0((\widetilde{134})_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 J_2^{(2)}(\widetilde{p_{134}}, p_2) \right. \\
& \quad \left. + \frac{1}{2} \left(\tilde{E}_4^0(2_{\bar{Q}}, 4_{\bar{q}}, 3_q, \hat{i}_g) - A_3^0(2_{\bar{Q}}, \hat{i}_g, 3_q) E_3^0((\widetilde{23})_{\bar{Q}}, 4_{\bar{q}}, \hat{i}_{\bar{q}}) \right. \right. \\
& \quad \left. \left. - A_3^0(2_{\bar{Q}}, \hat{i}_g, 4_{\bar{q}}) E_3^0((\widetilde{24})_Q, 3_q, \hat{i}_q) \right) |\mathcal{M}_4^0(1_Q, \hat{i}_g, \hat{j}_g, (\widetilde{234})_{\bar{Q}})|^2 J_2^{(2)}(p_1, \widetilde{p_{234}}) \right] \left. \right\}. \tag{5.23}
\end{aligned}$$

Since the \tilde{E}_4^0 antenna does not possess any soft $q\bar{q}$ limit, the subtraction term given in eq.(5.23) does not have any soft $q\bar{q}$ limits either and only subtracts the abelian triple collinear limit, as required. The sum $d\hat{\sigma}^{S,b(1)} + d\hat{\sigma}^{S,b(2)}$ will therefore correctly subtract both double soft and triple collinear limits of the double real radiation matrix element squared, without introducing any spurious single unresolved singularities.

5.5 Construction of the $d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,a}$ and $d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,d}$ subtraction terms

The $d\hat{\sigma}^{S,a}$ subtraction terms are NLO like: They are constructed as products of three-parton tree-level antennae and five-parton reduced matrix elements, and they subtract the single unresolved limits of the real radiation matrix elements. Thus, for the present calculation, they could a priori be taken over from [78], where we derived the NLO subtraction terms for $t\bar{t} + jet$ production.

However, at NNLO, the five-parton reduced matrix elements with remapped momenta present in these NLO-like subtraction terms develop further single unresolved limits. Those singularities are unphysical, since they do not correspond to any unresolved behaviour of the real radiation matrix element squared and must be cancelled by the $X_3^0 \cdot X_3^0$ pieces of the b-type subtraction terms. In $d\hat{\sigma}^{S,b}$, these $X_3^0 \cdot X_3^0$ terms are dictated by the requirement that all single unresolved limits of the four-parton antennae should be removed. As the choice of these four-parton antennae is fixed by the double unresolved behaviour of the real radiation matrix element, we do not have the freedom to choose the three-parton antenna functions

that make $d\hat{\sigma}^{S,a}$ simplest, as we did in [78]. Instead, these three-parton antennae have to be carefully chosen in such a way that the cancelation of the aforementioned unphysical singularities is achieved. With these considerations we obtain the $d\hat{\sigma}^{S,a}$ subtraction term as

$$\begin{aligned}
d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^{S,a} = & \mathcal{N}_{LO} N_F \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2} d\Phi_4(p_1, p_2, p_3, p_4; p_5, p_6) \sum_{(i,j) \in P(5,6)} \left\{ \right. \\
& N_c^2 \left[G_3^0(\hat{i}_g, 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, (\widetilde{34})_g, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, \hat{i}_g, (\widetilde{34})_g, 2_{\bar{Q}})|^2 \right. \right. \\
& \quad \left. \left. + (1/2) (|\mathcal{M}_5^0(1_Q, \hat{i}_g, (\widetilde{34})_g, \hat{j}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, (\widetilde{34})_g, \hat{i}_g, 2_{\bar{Q}})|^2) \right) J_2^{(3)}(p_1, p_2, \widetilde{p_{34}}) \right. \\
& \quad + A_3^0(2_{\bar{Q}}, \hat{i}_g, 3_q) |\mathcal{M}_5^0(1_Q, \hat{j}_g, 4_{\bar{q}}; ; \hat{i}_{\bar{q}}, (\widetilde{23})_{\bar{Q}})|^2 J_2^{(3)}(p_1, \widetilde{p_{23}}, p_4) \\
& \quad + a_3^0(3_q, \hat{i}_g, 4_{\bar{q}}) |\mathcal{M}_5^0(1_Q, (\widetilde{34})_{\bar{q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, 2_{\bar{Q}})|^2 J_2^{(3)}(p_1, p_2, \widetilde{p_{34}}) \\
& \quad + A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) |\mathcal{M}_5^0((\widetilde{14})_Q, \hat{i}_q; ; 3_q, \hat{j}_g, 2_{\bar{Q}})|^2 J_2^{(3)}(\widetilde{p_{14}}, p_2, p_3) \\
& \quad \left. \left. + a_3^0(4_{\bar{q}}, \hat{i}_g, 3_q) |\mathcal{M}_5^0(1_Q, \hat{j}_g, \hat{i}_q; ; (\widetilde{34})_q, 2_{\bar{Q}})|^2 J_2^{(3)}(p_1, p_2, \widetilde{p_{34}}) \right] \right. \\
& - G_3^0(\hat{i}_g, 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, (\widetilde{34})_g, \hat{i}_g, \hat{j}_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{i}_g, (\widetilde{34})_g, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \right) J_2^{(3)}(p_1, p_2, \widetilde{p_{34}}) \\
& - \frac{1}{2} E_3^0(1_Q, 3_q, 4_{\bar{q}}) |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{i}_g, \hat{j}_g, (\widetilde{34})_\gamma, 2_{\bar{Q}})|^2 J_2^{(3)}(\widetilde{p_{13}}, p_2, \widetilde{p_{34}}) \\
& - \frac{1}{2} E_3^0(2_{\bar{Q}}, 3_q, 4_{\bar{q}}) |\mathcal{M}_5^0(1_Q, \hat{i}_g, \hat{j}_g, (\widetilde{34})_\gamma, (\widetilde{23})_{\bar{Q}})|^2 J_2^{(3)}(p_1, \widetilde{p_{23}}, \widetilde{p_{34}}) \\
& - \frac{1}{2} A_3^0(1_Q, \hat{i}_g, 3_q) \left(|\mathcal{M}_5^0((\widetilde{13})_Q, \hat{j}_g, 4_{\bar{q}}; ; \hat{i}_{\bar{q}}, 2_{\bar{Q}})|^2 \right. \\
& \quad \left. + |\mathcal{M}_5^0((\widetilde{13})_Q, 4_{\bar{q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, 2_{\bar{Q}})|^2 \right) J_2^{(3)}(\widetilde{p_{13}}, p_2, p_4) \\
& - \frac{1}{2} A_3^0(2_{\bar{Q}}, \hat{i}_g, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{i}_q; ; 3_q, \hat{j}_g, (\widetilde{24})_{\bar{Q}})|^2 \right. \\
& \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{j}_g, \hat{i}_q; ; 3_q, (\widetilde{24})_{\bar{Q}})|^2 \right) J_2^{(3)}(p_1, \widetilde{p_{23}}, p_3) \\
& + A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) \left(|\mathcal{M}_5^0((\widetilde{14})_Q, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, \hat{i}_q)|^2 + |\mathcal{M}_5^0((\widetilde{14})_Q, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, \hat{i}_q)|^2 \right. \\
& \quad - 2 |\mathcal{M}_5^0((\widetilde{14})_Q, 2_{\bar{Q}}, 3_q, \hat{i}_q, \hat{j}_\gamma)|^2 - (1/2) |\mathcal{M}_5^0((\widetilde{14})_Q, \hat{i}_q; ; 3_q, \hat{j}_g, 2_{\bar{Q}})|^2 \\
& \quad \left. - (1/2) |\mathcal{M}_5^0((\widetilde{14})_Q, \hat{j}_g, \hat{i}_q; ; 3_q, 2_{\bar{Q}})|^2 \right) J_2^{(3)}(\widetilde{p_{14}}, p_2, p_3) \\
& + A_3^0(2_Q, \hat{i}_g, 3_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, (\widetilde{23})_{\bar{Q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, 4_{\bar{q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, (\widetilde{23})_{\bar{Q}}; ; \hat{i}_{\bar{q}}, 4_{\bar{q}})|^2 \right. \\
& \quad - 2 |\mathcal{M}_5^0(1_Q, (\widetilde{23})_{\bar{Q}}, \hat{i}_{\bar{q}}, 4_{\bar{q}}, \hat{j}_\gamma)|^2 - (1/2) |\mathcal{M}_5^0((1_Q, 4_{\bar{q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, (\widetilde{23})_{\bar{Q}})|^2 \\
& \quad \left. - (1/2) |\mathcal{M}_5^0((1_Q, \hat{j}_g, 4_{\bar{q}}; ; \hat{i}_{\bar{q}}, (\widetilde{23})_{\bar{Q}})|^2 \right) J_2^{(3)}(p_1, \widetilde{p_{23}}, p_4) \\
& + \frac{1}{N_c^2} \left[\frac{1}{2} E_3^0(1_Q, 3_q, 4_{\bar{q}}) |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{i}_\gamma, \hat{j}_\gamma, (\widetilde{34})_\gamma, 2_{\bar{Q}})|^2 J_2^{(3)}(\widetilde{p_{13}}, p_2, \widetilde{p_{34}}) \right. \\
& \quad + \frac{1}{2} E_3^0(2_{\bar{Q}}, 3_q, 4_{\bar{q}}) |\mathcal{M}_5^0(1_Q, \hat{i}_\gamma, \hat{j}_\gamma, (\widetilde{34})_\gamma, (\widetilde{23})_{\bar{Q}})|^2 J_2^{(3)}(p_1, \widetilde{p_{23}}, \widetilde{p_{34}}) \\
& \quad \left. - \frac{1}{2} A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) \left(|\mathcal{M}_5^0((\widetilde{14})_Q, 2_{\bar{Q}}; ; 3_q, \hat{j}_g, \hat{i}_q)|^2 + |\mathcal{M}_5^0((\widetilde{14})_Q, \hat{j}_g, 2_{\bar{Q}}; ; 3_q, \hat{i}_q)|^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -2|\mathcal{M}_5^0((\widetilde{14})_Q, 2_{\bar{Q}}, 3_q, \hat{i}_q, \hat{j}_\gamma)|^2 \Big) J_2^{(3)}(\widetilde{p_{14}}, p_2, p_3) \\
& -\frac{1}{2}A_3^0(1_Q, \hat{i}_g, 3_q) \Big(|\mathcal{M}_5^0((\widetilde{13})_Q, 2_{\bar{Q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, 4_{\bar{q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{j}_g, 2_{\bar{Q}}; ; \hat{i}_{\bar{q}}, 4_{\bar{q}})|^2 \\
& -2|\mathcal{M}_5^0((\widetilde{13})_Q, 2_{\bar{Q}}, \hat{i}_{\bar{q}}, 4_{\bar{q}}, \hat{j}_\gamma)|^2 \Big) J_2^{(3)}(\widetilde{p_{13}}, p_2, p_4) \\
& -\frac{1}{2}A_3^0(2_{\bar{Q}}, \hat{i}_g, 3_q) \Big(|\mathcal{M}_5^0(1_Q, (\widetilde{23})_{\bar{Q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, 4_{\bar{q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, (\widetilde{23})_{\bar{Q}}; ; \hat{i}_{\bar{q}}, 4_{\bar{q}})|^2 \\
& -2|\mathcal{M}_5^0(1_Q, (\widetilde{23})_{\bar{Q}}, \hat{i}_{\bar{q}}, 4_{\bar{q}}, \hat{j}_\gamma)|^2 \Big) J_2^{(3)}(\widetilde{p_{13}}, p_2, p_4) \\
& -\frac{1}{2}A_3^0(2_{\bar{Q}}, \hat{i}_g, 4_{\bar{q}}) \Big(|\mathcal{M}_5^0(1_Q, (\widetilde{24})_{\bar{Q}}; ; 3_q, \hat{j}_g, \hat{i}_q)|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, (\widetilde{24})_{\bar{Q}}; ; 3_q, \hat{i}_q)|^2 \\
& -2|\mathcal{M}_5^0(1_Q, (\widetilde{24})_{\bar{Q}}, 3_q, \hat{i}_q, \hat{j}_\gamma)|^2 \Big) J_2^{(3)}(p_1, \widetilde{p_{24}}, p_3) \Big] \Big\}.
\end{aligned} \tag{5.24}$$

Finally, the $d\hat{\sigma}^{S,d}$ terms are constructed in the usual fashion. They capture the double unresolved behaviour of the real matrix element squared when two colour-unconnected unresolved partons are present and they remove the double counting of double unresolved limits in $d\hat{\sigma}^{S,a}$. They are given by

$$\begin{aligned}
d\hat{\sigma}_{g\bar{g} \rightarrow Q\bar{Q}q\bar{q}}^{S,d} &= -\mathcal{N}_{LO} N_F \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2} d\Phi_4(p_1, p_2, p_3, p_4; p_5, p_6) \times \\
& \sum_{(i,j) \in P(5,6)} \left\{ N_c^2 A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) A_3^0(2_{\bar{Q}}, \hat{j}_g, 3_q) |\mathcal{M}_4^0((\widetilde{14})_Q, (\widetilde{23})_{\bar{Q}}, \hat{j}_{\bar{q}}, \hat{i}_q)|^2 J_2^{(2)}(\widetilde{p_{14}}, \widetilde{p_{23}}) \right. \\
& -\frac{3}{2} A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) A_3^0(2_{\bar{Q}}, \hat{j}_g, 3_q) |\mathcal{M}_4^0((\widetilde{14})_Q, (\widetilde{23})_{\bar{Q}}, \hat{j}_{\bar{q}}, \hat{i}_q)|^2 J_2^{(2)}(\widetilde{p_{14}}, \widetilde{p_{23}}) \\
& -\frac{1}{2} A_3^0(1_Q, \hat{i}_g, 3_q) A_3^0(2_{\bar{Q}}, \hat{i}_g, 4_{\bar{q}}) |\mathcal{M}_4^0((\widetilde{13})_Q, (\widetilde{24})_{\bar{Q}}, \hat{i}_{\bar{q}}, \hat{j}_q)|^2 J_2^{(2)}(\widetilde{p_{13}}, \widetilde{p_{24}}) \\
& +\frac{1}{N_c^2} \left[\frac{1}{2} A_3^0(1_Q, \hat{i}_g, 4_{\bar{q}}) A_3^0(2_{\bar{Q}}, \hat{j}_g, 3_q) |\mathcal{M}_4^0((\widetilde{14})_Q, (\widetilde{23})_{\bar{Q}}, \hat{j}_{\bar{q}}, \hat{i}_q)|^2 J_2^{(2)}(\widetilde{p_{14}}, \widetilde{p_{23}}) \right. \\
& \left. \left. +\frac{1}{2} A_3^0(1_Q, \hat{i}_g, 3_q) A_3^0(2_{\bar{Q}}, \hat{i}_g, 4_{\bar{q}}) |\mathcal{M}_4^0((\widetilde{13})_Q, (\widetilde{24})_{\bar{Q}}, \hat{i}_{\bar{q}}, \hat{j}_q)|^2 J_2^{(2)}(\widetilde{p_{13}}, \widetilde{p_{24}}) \right] \right\}.
\end{aligned} \tag{5.25}$$

6. Numerical results

To verify how well the subtraction terms approximate the double real contributions related to the partonic process $gg \rightarrow Q\bar{Q}q\bar{q}$, we have used **RAMBO** [96] to generate phase space points in the vicinity of the singular regions and computed the ratio

$$R = \frac{d\hat{\sigma}_{NNLO}^{RR}}{d\hat{\sigma}_{NNLO}^S} \tag{6.1}$$

for each of these points. As before, $d\hat{\sigma}_{NNLO}^{RR}$ stands for the double real radiation contributions while $d\hat{\sigma}_{NNLO}^S$ is the corresponding subtraction term. In each unresolved limit we define a control variable x that allows us to vary the proximity of the phase space points

to the singularity. For the difference $d\hat{\sigma}_{NNLO}^{RR} - d\hat{\sigma}_{NNLO}^S$ to be finite and numerically integrable in four dimensions, the ratio R should approach unity as we get close to any singularity. The phase space points were generated with a fixed centre-of-mass energy of $\sqrt{s} = 1000$ GeV, the heavy fermions were given a mass of 174.3 GeV, and the two hard jets were required to have $p_T > 50$ GeV.

6.1 Double soft limit

The double soft phase space configurations are characterised by the $Q\bar{Q}$ pair taking nearly the full center of mass energy of the event s , leaving the massless final state $q\bar{q}$ pair with almost zero energy, as depicted in fig.1(a). In fig.1(b) we show the ratio between the

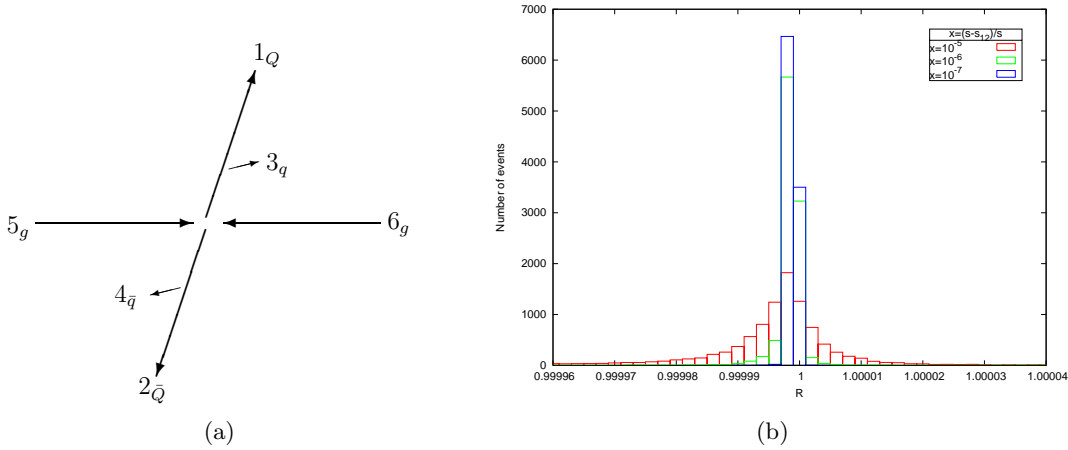


Figure 1: (a) Illustration of a double soft $q\bar{q}$ event. (b) Distribution of R for 10000 double soft phase space points.

double real radiation matrix element and the subtraction term for three different values of the control variable x defined in this case by, $x = (s - s_{12})/s$. It can be seen that as the soft $q\bar{q}$ pair, takes a smaller share of the total energy, i.e. as x becomes smaller, the peak of the distribution around $R = 1$ is sharper. This is a sign that the approximation improves as the limit is approached.

6.2 Triple collinear limit

Other double unresolved configurations in which the amplitude for this process is singular are the triple collinear limits $\hat{i}_g||3_q||4_{\bar{q}}$ illustrated in fig.2(a). In fig.2(b) we show how, as we make the control variable $x = -s_{345}/s$ smaller, that is, as we get closer in phase space to the singularity $\hat{5}_g||3_q||4_{\bar{q}}$ of the real radiation matrix element squared, the histogram becomes more pronouncedly peaked around unity. This signals us again that the approximation is good. Similar results which are not shown are obtained for the triple collinear limit involving the other initial state gluon $\hat{6}_g$.

6.3 Double collinear limits

The last type of double unresolved limits that the amplitude for the process $gg \rightarrow Q\bar{Q}q\bar{q}$ has, are the initial-final double collinear limits $\hat{i}_g||3_q + \hat{j}_g||4_{\bar{q}}$. In fig. 3(a) we represent

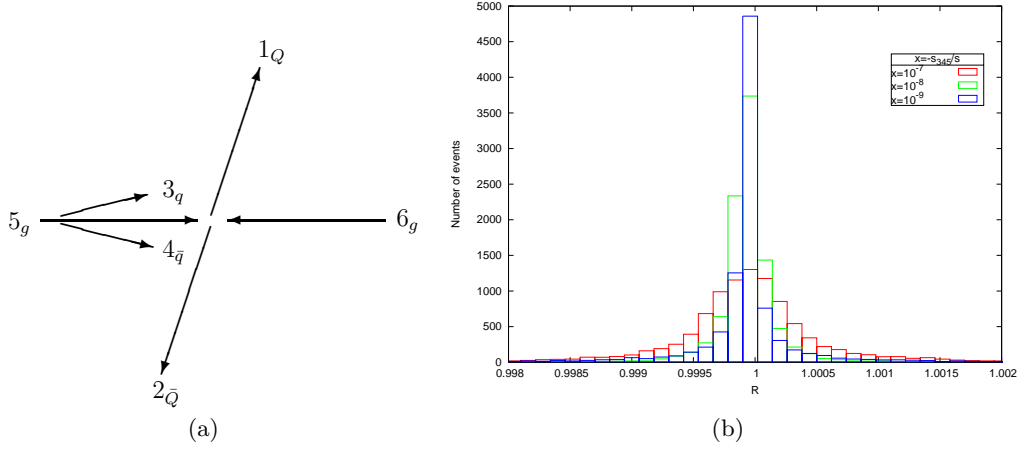


Figure 2: (a) Illustration of a triple collinear event. (b) Distribution of R for 10000 triple collinear phase space points.

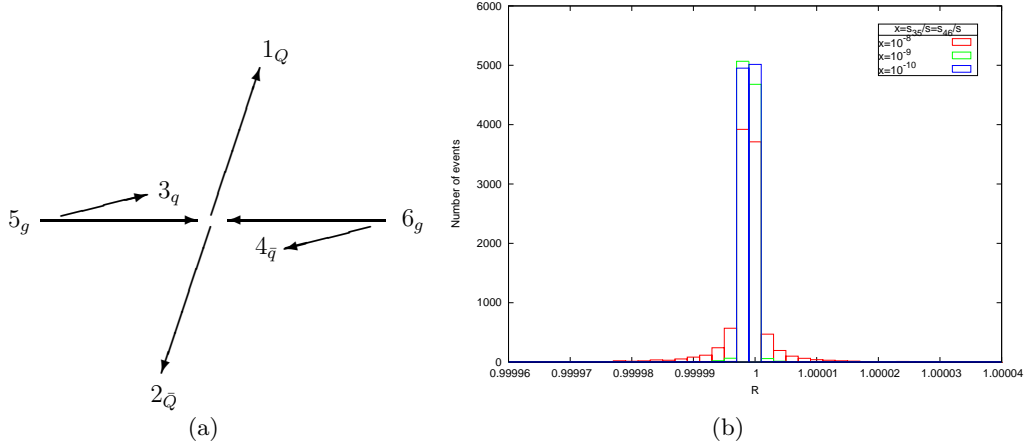


Figure 3: (a) Illustration of a double collinear event. (b) Distribution of R for 10000 double collinear phase space points.

schematically the kinematical configuration where $\hat{5}_g || 3_q + \hat{6}_g || 4_{\bar{q}}$. As it can be seen in fig. 3(b), the convergence of the subtraction term to the real radiation matrix elements is indeed achieved as we make the control variable smaller. The same results are obtained for the double collinear limit $\hat{6}_g || 3_q + \hat{5}_g || 4_{\bar{q}}$.

6.4 Final-final single collinear limit

In addition to the double unresolved limits discussed above, the double real radiation matrix element contains two types of single collinear limits. For the final-final collinear limit $3_q || 4_{\bar{q}}$ depicted in fig.4(a) we obtain the results shown in fig.4(b), where, once more, a good convergence of the subtraction terms to the real radiation contributions is achieved as the limit is approached. It should be noted that due to the presence of angular correlation terms this good convergence is only achieved after integration over the azimuthal angle of the collinear pair. The procedure by which these angular correlations are integrated out

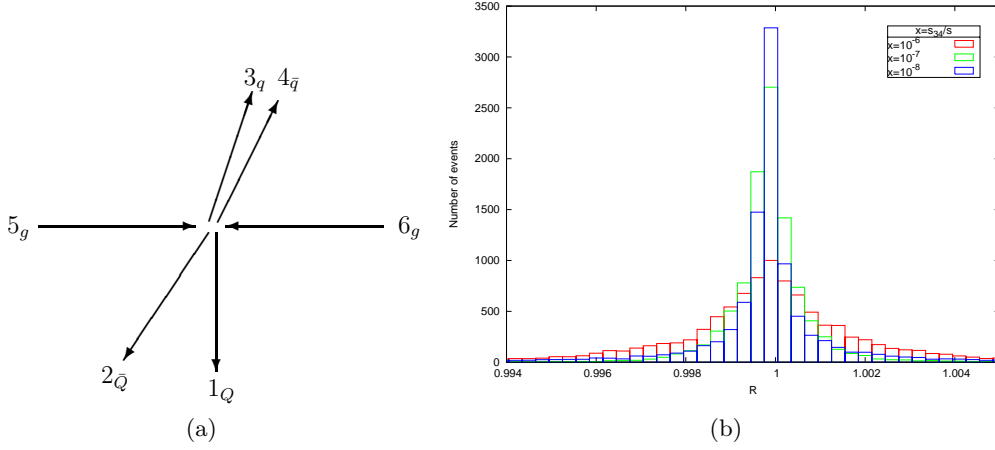


Figure 4: (a) Illustration of a final-final single collinear event. (b) Distribution of R for 10000 final-final single collinear phase space points.

has been explained in the case where massive partons are present in [69].

6.5 Initial-final single collinear limits

The last type of unresolved limits are the four collinear limits between the massless final state (anti) quark and one of the initial state gluons. As it can be seen in fig.5(b), also in

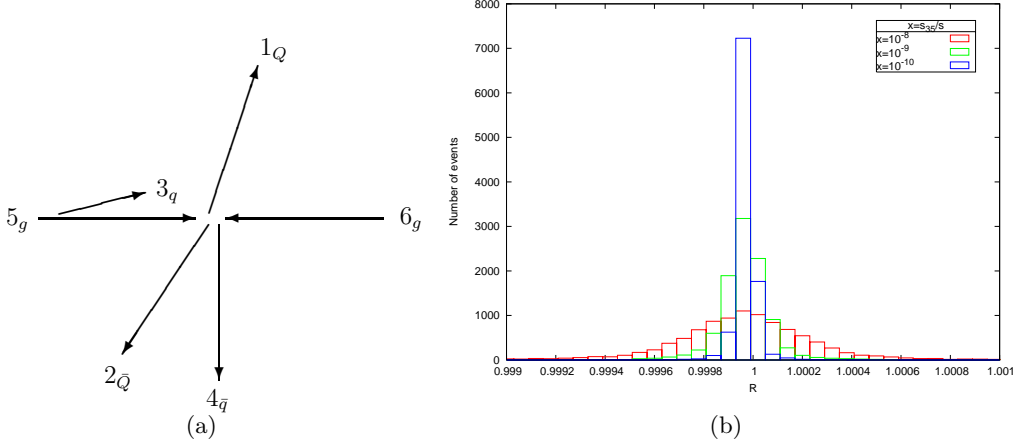


Figure 5: (a) Illustration of an initial-final single collinear event. (b) Distribution of R for 10000 initial-final single collinear phase space points.

these limits our subtraction terms constitute a valid approximation of the real radiation differential cross section. As it is indicated in fig.5(a) the histogram in fig.5(b) corresponds to the limit $\hat{5}_g \parallel 3_q$. Similar results are also obtained for the other three initial-final single collinear limits: $\hat{6}_g \parallel 3_q$, $\hat{5}_g \parallel 4_{\bar{q}}$, and $\hat{6}_g \parallel 4_{\bar{q}}$.

7. Conclusions

In this paper, we present the double real radiation contributions to the $t\bar{t}$ hadronic pro-

duction cross section coming from the partonic process $gg \rightarrow t\bar{t}q\bar{q}$. These contributions develop infrared divergencies when one or two partons become unresolved (soft or collinear) such that a systematic subtraction procedure is required before these contributions can be evaluated numerically.

We follow the formalism developed in [69], where we extended the NNLO antenna subtraction formalism to include the evaluation of hadronic observables involving a massive pair of fermions. Section 2 contains a summary of the method used in this paper.

To capture the double unresolved singular behaviour of the gluon-gluon initiated matrix elements, we derive the appropriate four-parton antenna functions and establish their limiting behaviour respectively in sections 3 and 4. Using these, in section 5, we explicitly construct the antenna subtraction terms for the $gg \rightarrow t\bar{t}q\bar{q}$ subprocess in leading and subleading colour contributions.

In section 6, we check numerically that our subtraction terms approximate the real matrix elements in all single and double unresolved configurations in a point-by-point manner. In the regions of phase space associated to the single unresolved collinear kinematical configurations, the ratio between real matrix elements and subtraction term approaches unity, provided the azimuthal terms associated with these collinear limits are correctly treated.

The double real corrections stemming from the partonic channel $gg \rightarrow t\bar{t}q\bar{q}$ and contributing to the hadronic production of a top-antitop pair presented here provide a substantial step towards the calculation of the NNLO corrections to the top-quark pair production at the LHC. Future steps include in particular the computation of the remaining double real subtraction terms related to partonic channels involving only gluons in initial and final state and the computation of mixed real-virtual contributions for all partonic channels involved.

8. Acknowledgements

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A. Three-parton antennae

In this appendix we list the three-parton antenna functions used in the subtraction terms of section 5 together with their single unresolved limits. The massive and massless antennae can be found in [78, 80] and [67, 72] respectively, together with their integrated forms. All corresponding single unresolved factors can be found in section 4.1. The massive A-type flavour violating initial-final antenna initiated by a gluon, is new. It will be given below in unintegrated and integrated form together with its infrared limits.

In order to make the mass-dependence in the expressions of the antenna functions explicit, we use the same convention as in the paper and define our invariants as $s_{ij} = 2p_i \cdot p_j$.

A.1 Massive final-final antennae

Our subtraction terms use A and E-type massive final-final antennae. The former is given by

$$\begin{aligned}
A_3^0(1_Q, 3_g, 2_{\bar{Q}}) = & \frac{1}{(E_{cm}^2 + 2m_Q^2)} \left(\frac{2s_{12}^2}{s_{13}s_{23}} + \frac{2s_{12}}{s_{13}} + \frac{2s_{12}}{s_{23}} + \frac{s_{23}}{s_{13}} + \frac{s_{13}}{s_{23}} \right. \\
& + m_Q^2 \left(\frac{8s_{12}}{s_{13}s_{23}} - \frac{2s_{12}}{s_{13}^2} - \frac{2s_{12}}{s_{23}^2} - \frac{2s_{23}}{s_{13}^2} - \frac{2}{s_{13}} - \frac{2}{s_{23}} - \frac{2s_{13}}{s_{23}^2} \right) \\
& \left. + m_Q^4 \left(-\frac{8}{s_{23}^2} - \frac{8}{s_{13}^2} \right) \right) + \mathcal{O}(\epsilon), \tag{A.1}
\end{aligned}$$

with $E_{cm}^2 = (p_1 + p_2 + p_3)^2$. It has only a single soft limit:

$$A_3^0(1_Q, 3_g, 2_{\bar{Q}}) \xrightarrow{p_3 \rightarrow 0} \mathcal{S}_{132}(m_Q, m_Q). \tag{A.2}$$

This antenna has been derived and integrated in [80].

The E-type antenna is

$$E_3^0(1_Q, 3_q, 4_{\bar{q}}) = \frac{1}{(E_{cm}^2 - m_Q^2)^2} \left(s_{13} + s_{14} + \frac{s_{13}^2}{s_{34}} + \frac{s_{14}^2}{s_{34}} - 2E_{cm}m_Q \right) + \mathcal{O}(\epsilon), \tag{A.3}$$

where $E_{cm}^2 = (p_1 + p_3 + p_4)^2$. Its only infrared limit is

$$E_3^0(1_Q, 3_q, 4_{\bar{q}}) \xrightarrow{3_q || 4_{\bar{q}}} \frac{1}{s_{34}} P_{q\bar{q} \rightarrow G}(z). \tag{A.4}$$

This antenna has been derived and integrated in [78].

A.2 Massless initial-final antennae

We use the following A-type antenna

$$a_3^0(1_q, \hat{3}_g, 2_{\bar{q}}) = \frac{1}{s_{123}(s_{13} + s_{23})} \left(-\frac{2s_{12}^2}{s_{13}} + \frac{2s_{23}s_{12}}{s_{13}} + 2s_{12} - \frac{s_{23}^2}{s_{13}} - s_{23} \right) + \mathcal{O}(\epsilon), \tag{A.5}$$

with $s_{123} = s_{12} - s_{13} - s_{23}$. It is obtained by partial fractioning the full antenna $A_3^0(1_q, \hat{3}_g, 2_{\bar{q}})$ so that it only contains the following single collinear limit

$$a_3^0(1_q, \hat{3}_g, 2_{\bar{q}}) \xrightarrow{1_q || 3_g} \frac{1}{s_{13}} P_{qg \leftarrow Q}(z). \tag{A.6}$$

We also need the following G-type antenna

$$G_3^0(\hat{1}_g, 3_q, 4_{\bar{q}}) = \frac{1}{s_{134}^2} \left(\frac{s_{13}^2}{s_{34}} + \frac{s_{14}^2}{s_{34}} \right) + \mathcal{O}(\epsilon) \tag{A.7}$$

which has the single collinear limit

$$G_3^0(\hat{1}_g, 3_q, 4_{\bar{q}}) \xrightarrow{3_q || 4_{\bar{q}}} \frac{1}{s_{34}} P_{q\bar{q} \rightarrow G}(z). \tag{A.8}$$

All massless initial-final antennae have been presented in unintegrated and in integrated forms in [72].

A.3 Massive initial-final antennae

Initial-Final E and D-type antennae

The massive initial-final three-parton D-type antenna is given by

$$\begin{aligned}
D_3^0(1_Q, 3_g, \hat{4}_g) = & \frac{1}{(Q^2 + m_Q^2)^2} \left(9s_{13} - 9s_{14} - 6s_{34} + \frac{4s_{14}^2}{s_{13}} + \frac{s_{34}^2}{s_{13}} - \frac{s_{34}^2}{s_{14}} + \frac{3s_{14}s_{34}}{s_{13}} \right. \\
& + \frac{3s_{13}s_{34}}{s_{14}} - \frac{4s_{13}^2}{s_{14}} + \frac{2s_{14}^3}{s_{13}s_{34}} - \frac{4s_{13}^2}{s_{34}} - \frac{4s_{14}^2}{s_{34}} + \frac{6s_{13}s_{14}}{s_{34}} + \frac{2s_{13}^3}{s_{14}s_{34}} + 2m_Q m_\chi \\
& + m_Q^2 \left(-\frac{2s_{13}^2}{s_{14}^2} + \frac{4s_{34}s_{13}}{s_{14}^2} + \frac{4s_{13}}{s_{14}} - \frac{2s_{34}^2}{s_{14}^2} - \frac{6s_{34}}{s_{14}} + \frac{2s_{34}^2}{s_{14}s_{13}} + \frac{4s_{14}}{s_{13}} \right. \\
& \left. \left. + \frac{6s_{34}}{s_{13}} - \frac{2s_{14}^2}{s_{13}^2} - \frac{2s_{34}^2}{s_{13}^2} - \frac{4s_{14}s_{34}}{s_{13}^2} - 6 \right) - \frac{2m_Q^3 m_\chi s_{34}}{s_{13}s_{14}} + \frac{2m^4 s_{34}}{s_{13}s_{14}} \right) + \mathcal{O}(\epsilon),
\end{aligned} \tag{A.9}$$

where $Q^2 = -(p_1 + p_3 - p_4)^2$ and $m_\chi = \sqrt{Q^2}$. It has a soft gluon limit as well as a single collinear limit

$$D_3^0(1_Q, 3_g, \hat{4}_g) \xrightarrow{p_3 \rightarrow 0} \mathcal{S}_{134}(m_Q, 0), \tag{A.10}$$

$$D_3^0(1_Q, 3_g, \hat{4}_g) \xrightarrow{3_g || 4_g} \frac{1}{s_{34}} P_{gg \leftarrow G}(z). \tag{A.11}$$

This antenna is related to the split antennae $D_3^0(4_g; 3_g, 1_Q)$ and $D_3^0(4_g; 1_Q, 3_g)$ defined in [78] through

$$D_3^0(1_Q, 3_g, \hat{4}_g) = D_3^0(4_g; 3_g, 1_Q) - D_3^0(4_g; 1_Q, 3_g). \tag{A.12}$$

The massive initial-final E-type antenna used in our subtraction terms reads

$$E_3^0(1_Q, 3_q, \hat{4}_q) = -\frac{1}{(Q^2 + m_Q^2)^2} \left(-s_{14} + s_{13} - \frac{s_{13}^2}{s_{34}} - \frac{s_{14}^2}{s_{34}} - 2m_Q m_\chi \right) + \mathcal{O}(\epsilon), \tag{A.13}$$

with $Q^2 = -(p_1 + p_3 - p_4)^2$, and $m_\chi = \sqrt{Q^2}$. The only infrared limit that this antenna has is

$$E_3^0(1_Q, 3_q, \hat{4}_q) \xrightarrow{3_q || 4_q} \frac{1}{s_{34}} P_{qq \leftarrow Q}(z). \tag{A.14}$$

Both of these antennae have been derived and integrated in [78].

Massive flavour violating initial-final A-type antenna

The new three-parton flavour violating A-type antenna initiated by a gluon is denoted by $A_3^0(1_Q, \hat{3}_g, 2_{\bar{q}})$ and reads

$$A_3^0(1_Q, \hat{3}_g, 2_{\bar{q}}) = \frac{1}{(Q^2 + m_Q^2)} \left(\frac{2s_{12}^2}{s_{13}s_{23}} - \frac{2s_{12}}{s_{13}} - \frac{2s_{12}}{s_{23}} + \frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} \right)$$

$$-m_Q^2 \left(\frac{2s_{12}}{s_{13}^2} - \frac{2s_{23}}{s_{13}^2} - \frac{2}{s_{13}} \right) + \mathcal{O}(\epsilon), \quad (\text{A.15})$$

where $Q^2 = -(p_1 + p_2 - p_3)^2$.

Its infrared limit is

$$A_3^0(1_Q, \hat{3}_g, 2_{\bar{q}}) \xrightarrow{2_{\bar{q}} || \hat{3}_g} \frac{1}{s_{23}} P_{qg \leftarrow Q}(z). \quad (\text{A.16})$$

Its integrated form denoted by $\mathcal{A}_3^0(1_Q, \hat{3}_g, 2_{\bar{q}})$ is obtained by integrating the expression given in eq.(A.15) over the unresolved initial-final antenna phase space involving one massive parton of mass m_Q [78]. The integrated antenna is given by,

$$\mathcal{A}_3^0(1_Q, \hat{3}_g, 2_{\bar{q}}) = \frac{1}{C(\epsilon)} \int d\Phi_2 \frac{(Q^2 + m_K^2)}{2\pi} A_3^0(1_Q, \hat{3}_g, 2_{\bar{q}}) \quad (\text{A.17})$$

where $C(\epsilon)$, is a normalisation factor given by

$$C(\epsilon) = (4\pi)^\epsilon \frac{e^{-\epsilon\gamma_E}}{8\pi^2} \quad (\text{A.18})$$

and the initial-final massive antenna phase space denoted by $d\Phi_{X_{i,jk}}$ is given by,

$$d\Phi_{X_{i,jk}} = d\Phi_2 \frac{(Q^2 + m_K^2)}{2\pi}. \quad (\text{A.19})$$

In eq.(A.19), $d\Phi_2$ is the massive $2 \rightarrow 2$ phase space involving one massive final state particle, which we parametrise as [78]

$$d\phi_2(m_Q, 0) = \frac{(4\pi)^{\epsilon-1}}{2\Gamma(1-\epsilon)} E_{cm}^{2\epsilon-2} (E_{cm}^2 - m_Q^2)^{1-2\epsilon} dy y^{-\epsilon} (1-y)^{-\epsilon}, \quad (\text{A.20})$$

where y runs from 0 to 1. In this context, the invariants are given by,

$$2p_i \cdot p_j = \frac{Q^2 + m_Q^2}{xE_{cm}^2} [E_{cm}^2 - y(E_{cm}^2 - m_Q^2)] \quad (\text{A.21})$$

$$2p_i \cdot p_k = \frac{Q^2 + m_Q^2}{xE_{cm}^2} [y(E_{cm}^2 - m_Q^2)] \quad (\text{A.22})$$

and the center of mass energy E_{cm} is

$$E_{cm} = \sqrt{\frac{Q^2(1-x) + m_Q^2}{x}} \quad \text{with} \quad x = \frac{Q^2 + m_j^2}{2p_i \cdot q}. \quad (\text{A.23})$$

The integrated antenna reads

$$\begin{aligned} \mathcal{A}_3^0(1_Q, \hat{3}_g, 2_{\bar{q}}) &= (Q^2 + m_Q^2)^{-\epsilon} \times \left[-\frac{1}{2\epsilon} p_{qg}^{(0)}(x) - \frac{1 - 4x + 2x^2 + xx_0 + 2x^2x_0 - 2x^3x_0}{2(1 - xx_0)} \right. \\ &\quad \left. + (1 - 2x + 2x^2) \left(\ln(1-x) + \frac{1}{2} \ln(x^2(1-x_0)) \right) \right] + \mathcal{O}(\epsilon) \end{aligned} \quad (\text{A.24})$$

with

$$x_0 = \frac{Q^2}{Q^2 + m_Q^2} \quad (\text{A.25})$$

and the splitting kernel $p_{qg}^{(0)}(x)$ given by

$$p_{qg}^{(0)}(x) = 1 - 2x + 2x^2. \quad (\text{A.26})$$

Having a single collinear limit in its unintegrated form, this integrated antenna develops a pole proportionnal to the splitting kernel $p_{qg}^{(0)}(x)$, as expected.

A.4 Massless initial-initial antennae

Our subtraction terms employ two types of three-parton initial-initial antennae: F and G-type antennae. The F-type antenna is given by

$$\begin{aligned} F_3^0(\hat{1}_g, 2_g, \hat{3}_g) = & \frac{1}{s_{123}^2} \left(-12s_{12} + 12s_{13} - 12s_{23} + \frac{2s_{12}^3}{s_{13}s_{23}} + \frac{4s_{12}^2}{s_{13}} - \frac{4s_{12}^2}{s_{23}} \right. \\ & + \frac{6s_{23}s_{12}}{s_{13}} + \frac{6s_{13}s_{12}}{s_{23}} + \frac{4s_{23}^2}{s_{13}} - \frac{4s_{13}^2}{s_{23}} + \frac{2s_{23}^3}{s_{13}s_{12}} - \frac{4s_{13}^2}{s_{12}} \\ & \left. - \frac{4s_{23}^2}{s_{12}} + \frac{6s_{13}s_{23}}{s_{12}} + \frac{2s_{13}^3}{s_{23}s_{12}} \right) + \mathcal{O}(\epsilon) \end{aligned} \quad (\text{A.27})$$

with $s_{123} = s_{13} - s_{12} - s_{23}$. Its infrared limits are

$$F_3^0(\hat{1}_g, 2_g, \hat{3}_g) \xrightarrow{p_2 \rightarrow 0} \mathcal{S}_{123}(0, 0), \quad (\text{A.28})$$

$$F_3^0(\hat{1}_g, 2_g, \hat{3}_g) \xrightarrow{\hat{1}_g || 2_g} \frac{1}{s_{12}} P_{gg \leftarrow G}(z), \quad (\text{A.29})$$

$$F_3^0(\hat{1}_g, 2_g, \hat{3}_g) \xrightarrow{2_g || \hat{3}_g} \frac{1}{s_{23}} P_{gg \leftarrow G}(z). \quad (\text{A.30})$$

Finally the initial-initial G-type antenna function is

$$G_3^0(\hat{1}_g, 3_q, \hat{4}_q) = \frac{1}{s_{134}^2} \left(-\frac{s_{13}^2}{s_{34}} - \frac{s_{14}^2}{s_{34}} \right) + \mathcal{O}(\epsilon). \quad (\text{A.31})$$

with, $s_{134} = s_{14} - s_{13} - s_{34}$. It only has a single initial-final collinear limit

$$G_3^0(\hat{1}_g, 3_q, \hat{4}_q) \xrightarrow{3_q || \hat{4}_q} \frac{1}{s_{34}} P_{gq \leftarrow Q}(z). \quad (\text{A.32})$$

Both of these antennae have been derived and presented in their unintegrated and integrated forms in [72].

B. Four-parton antennae

In this section, we list the two known four-parton antenna functions used to construct our subtraction term presented in section 5: the massive final-final B type antenna with a massive $Q\bar{Q}$ pair as radiators and the massless initial-initial G-type antenna with two initial state gluons as radiators. Their infrared limits were given in section 4.

B.1 Massive final-final B-type antennae

This antenna, denoted by $B_4^0(1_Q, 4_{\bar{q}}, 3_q, 2_{\bar{Q}})$, reads

$$\begin{aligned}
B_4^0(1_Q, 4_{\bar{q}}, 3_q, 2_{\bar{Q}}) = & \frac{1}{(E_{cm}^2 + 2m_Q^2)} \left\{ \frac{1}{s_{34}s_{134}^2} [s_{12}s_{13} + s_{12}s_{14} + s_{13}s_{23} + s_{14}s_{24}] \right. \\
& + \frac{1}{s_{34}s_{234}^2} [s_{12}s_{23} + s_{12}s_{24} + s_{13}s_{23} + s_{14}s_{24}] \\
& + \frac{1}{s_{34}^2 s_{134}^2} [2s_{12}s_{13}s_{14} + s_{13}s_{14}s_{24} + s_{13}s_{14}s_{23} - s_{13}^2 s_{24} - s_{14}^2 s_{23}] \\
& + \frac{1}{s_{34}^2 s_{234}^2} [2s_{12}s_{23}s_{24} + s_{13}s_{23}s_{24} + s_{14}s_{23}s_{24} - s_{13}s_{24}^2 - s_{14}s_{23}^2] \\
& + \frac{1}{s_{34}s_{134}s_{234}} [2s_{12}^2 + s_{12}s_{23} + s_{12}s_{24} + s_{12}s_{13} + s_{12}s_{14}] \\
& + \frac{1}{s_{34}^2 s_{134}s_{234}} [-s_{13}s_{24}^2 - s_{14}s_{23}^2 - s_{13}^2 s_{24} - s_{14}^2 s_{23} \\
& + s_{13}s_{14}s_{23} + s_{13}s_{14}s_{24} + s_{13}s_{23}s_{24} + s_{14}s_{23}s_{24} \\
& - 2s_{12}s_{13}s_{24} - 2s_{12}s_{14}s_{23}] + \frac{2s_{12}}{s_{134}s_{234}} \\
& + m_Q^2 \left[\frac{8s_{13}s_{14}}{s_{34}^2 s_{134}^2} + \frac{8s_{23}s_{24}}{s_{34}^2 s_{234}^2} - \frac{4}{s_{134}^2} - \frac{4}{s_{234}^2} \right. \\
& - \frac{2}{s_{34}s_{134}^2} [s_{12} + s_{23} + s_{24}] - \frac{2}{s_{34}s_{234}^2} [s_{12} + s_{13} + s_{14}] \\
& + \frac{2}{s_{34}s_{134}s_{234}} [4s_{12} - s_{13} - s_{14} - s_{23} - s_{24}] \\
& \left. - \frac{8}{s_{34}^2 s_{134}s_{234}} [s_{14}s_{23} + s_{13}s_{24}] \right] \\
& \left. - m_Q^4 \left[\frac{8}{s_{34}s_{134}^2} + \frac{8}{s_{34}s_{234}^2} \right] \right\} + \mathcal{O}(\epsilon), \tag{B.1}
\end{aligned}$$

where $E_{cm}^2 = (p_1 + p_2 + p_3 + p_4)^2$. It is normalised to the tree-level two-parton matrix element (with couplings and colour factors omitted)

$$|\mathcal{M}_2^0(\gamma^* \rightarrow Q\bar{Q})|^2 = 4 [(1 - \epsilon)E_{cm}^2 + 2m_Q^2]. \tag{B.2}$$

B.2 Massless initial-initial G-type antenna

This antenna denoted by $G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g)$ reads

$$\begin{aligned}
G_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, \hat{2}_g) = & \frac{1}{Q^4} \left\{ \frac{1}{s_{12}^2 s_{34}^2} [2s_{13}s_{14}s_{23}s_{24} - s_{13}^2 s_{24}^2 - s_{14}^2 s_{23}^2] + \frac{s_{23}}{s_{12}s_{13}s_{34}} [s_{23}^2 + s_{24}^2] \right. \\
& + \frac{s_{23}}{s_{12}s_{13}s_{134}} [s_{23}^2 + 2s_{24}s_{34} + s_{24}^2 + s_{34}^2] + \frac{s_{23}}{s_{12}s_{13}} [s_{14} + 2s_{24} + s_{34}] \\
& + \frac{s_{13}}{s_{12}} - \frac{1}{s_{12}s_{34}s_{134}} \left[-2s_{14}s_{23}^2 + 2s_{14}s_{24}^2 + 2s_{14}^2 s_{23} + 2s_{14}^2 s_{24} + s_{23}s_{24}^2 \right. \\
& \left. + s_{23}^2 s_{24} + s_{23}^3 + s_{24}^3 \right] + \frac{1}{s_{12}s_{34}} [2s_{13}s_{14} + 4s_{13}s_{23} + 3s_{13}s_{24} + 2s_{13}^2] \\
& \left. + \frac{1}{s_{12}s_{34}} [2s_{13}s_{14} + 4s_{13}s_{23} + 3s_{13}s_{24} + 2s_{13}^2] \right\}
\end{aligned}$$

$$\begin{aligned}
& -s_{14}s_{23} + 4s_{14}^2] + \frac{1}{s_{12}s_{134}} [2s_{14}s_{24} - 4s_{23}s_{24} - s_{23}s_{34} - s_{24}s_{34}] \\
& + \frac{s_{12}}{s_{34}s_{134}s_{234}} [-2s_{12}s_{14} + s_{12}^2 + 4s_{14}s_{24} + 4s_{14}^2 + 4s_{24}^2] \\
& + \frac{s_{12}}{s_{134}s_{234}} [-6s_{14} - 6s_{24} + 3s_{34}] + \frac{2s_{12}^2s_{14}s_{24}}{s_{34}^2s_{134}s_{234}} + \frac{1}{2s_{13}s_{24}} [s_{12}s_{34} \\
& - s_{14}s_{23}] + \frac{s_{24}}{s_{13}s_{34}s_{234}} [2s_{12}s_{24} + s_{12}^2 + 2s_{24}^2] - \frac{1}{s_{13}s_{34}} [2s_{12}s_{23} \\
& - 2s_{12}s_{24} - s_{12}^2 + 2s_{23}s_{24} - 2s_{23}^2 - 2s_{24}^2] - \frac{s_{34}}{s_{13}s_{134}^2} [2s_{12}s_{23} + 2s_{12}s_{24} \\
& - s_{12}^2 - 2s_{23}s_{24} - s_{23}^2 - s_{24}^2] - \frac{1}{s_{13}s_{134}s_{234}} [-6s_{12}s_{24}s_{34} + 4s_{12}s_{24}^2 \\
& + 3s_{12}s_{34}^2 + 3s_{12}^2s_{24} - 3s_{12}^2s_{34} - s_{12}^3 + 3s_{24}s_{34}^2 - 4s_{24}^2s_{34} + 2s_{24}^3 - s_{34}^3] \\
& - \frac{1}{s_{13}s_{134}} [-8s_{12}s_{23} + 4s_{12}s_{24} - 4s_{12}s_{34} + 4s_{12}^2 + 2s_{23}s_{34} + 6s_{23}^2 + 2s_{24}^2 \\
& + 2s_{34}^2] - \frac{1}{s_{13}s_{234}} [4s_{12}s_{24} - 2s_{12}s_{34} + s_{12}^2 - 3s_{24}s_{34} + 4s_{24}^2 + s_{34}^2] \\
& - \frac{1}{s_{13}} [2s_{12} - 2s_{23} + 2s_{24} - 2s_{34}] - \frac{s_{14}}{s_{34}^2s_{134}} [4s_{12}s_{14} - 8s_{12}s_{24} + 2s_{12}^2 \\
& + 4s_{14}s_{23} + 4s_{14}s_{24} - 4s_{23}s_{24} + 4s_{24}^2] + \frac{s_{14}^2}{s_{34}^2s_{134}^2} [4s_{12}s_{23} + 4s_{12}s_{24} \\
& - 2s_{12}^2 - 4s_{23}s_{24} - 2s_{23}^2 - 2s_{24}^2] + \frac{1}{s_{34}^2} [4s_{12}s_{13} - 4s_{12}s_{14} - s_{12}^2 \\
& + 4s_{13}s_{23} - 2s_{13}s_{24} - 2s_{13}^2 + 2s_{14}s_{23}] + \frac{1}{s_{34}s_{134}} [-4s_{12}s_{23} - 2s_{12}s_{24} \\
& + 5s_{12}^2 - 8s_{14}s_{23} + 6s_{14}s_{24} + 6s_{14}^2 + 6s_{23}^2 + 4s_{24}^2] + \frac{1}{s_{34}} [2s_{12} - 2s_{13} \\
& - 6s_{14}] + \frac{1}{s_{134}^2} [2s_{12}s_{23} + 2s_{12}s_{24} - s_{12}^2 - 2s_{23}s_{24} - s_{23}^2 - s_{24}^2] \\
& + \frac{1}{s_{134}} [4s_{12} - 4s_{14} + 4s_{23} + 2s_{24} + 3s_{34}] - \frac{7}{2} + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \Big\} \\
& + \mathcal{O}(\epsilon),
\end{aligned} \tag{B.3}$$

where $Q^2 = s_{12} + s_{34} - s_{13} - s_{14} - s_{23} - s_{24}$, $s_{134} = s_{34} - s_{13} - s_{14}$, and $s_{234} = s_{34} - s_{23} - s_{24}$. This antenna is normalised to the matrix element squared associated to the process $gg \rightarrow H$ which is given by

$$|\mathcal{M}_2^0(gg \rightarrow H)|^2 = \frac{1}{2}(1 - \epsilon)Q^4. \tag{B.4}$$

References

- [1] D0 Collaboration, S. Abachi *et. al.*, *Observation of the top quark*, *Phys.Rev.Lett.* **74** (1995) 2632–2637 [[hep-ex/9503003](#)].
- [2] CDF Collaboration, F. Abe *et. al.*, *Observation of top quark production in $\bar{p}p$ collisions*, *Phys.Rev.Lett.* **74** (1995) 2626–2631 [[hep-ex/9503002](#)].

- [3] ATLAS Collaboration, CMS Collaboration, P. Silva, f. t. A. Collaboration and f. t. C. Collaboration, *Recent results on Top quark Physics with the ATLAS and CMS experiments*, 1206.2967.
- [4] T. Kinoshita, *Mass singularities of Feynman amplitudes*, *J.Math.Phys.* **3** (1962) 650–677.
- [5] T. Lee and M. Nauenberg, *Degenerate Systems and Mass Singularities*, *Phys.Rev.* **133** (1964) B1549–B1562.
- [6] G. Bevilacqua, M. Czakon, A. van Hameren, C. G. Papadopoulos and M. Worek, *Complete off-shell effects in top quark pair hadroproduction with leptonic decay at next-to-leading order*, *JHEP* **1102** (2011) 083 [1012.4230].
- [7] A. Denner, S. Dittmaier, S. Kallweit and S. Pozzorini, *NLO QCD corrections to WWbb production at hadron colliders*, *Phys.Rev.Lett.* **106** (2011) 052001 [1012.3975].
- [8] P. Nason, S. Dawson and R. K. Ellis, *The One Particle Inclusive Differential Cross-Section for Heavy Quark Production in Hadronic Collisions*, *Nucl.Phys.* **B327** (1989) 49–92.
- [9] M. Cacciari, S. Frixione, M. L. Mangano, P. Nason and G. Ridolfi, *Updated predictions for the total production cross sections of top and of heavier quark pairs at the Tevatron and at the LHC*, *JHEP* **0809** (2008) 127 [0804.2800].
- [10] N. Kidonakis and R. Vogt, *The Theoretical top quark cross section at the Tevatron and the LHC*, *Phys.Rev.* **D78** (2008) 074005 [0805.3844].
- [11] S. Moch and P. Uwer, *Theoretical status and prospects for top-quark pair production at hadron colliders*, *Phys.Rev.* **D78** (2008) 034003 [0804.1476].
- [12] V. Ahrens, A. Ferroglia, M. Neubert, B. D. Pecjak and L. L. Yang, *Renormalization-Group Improved Predictions for Top-Quark Pair Production at Hadron Colliders*, *JHEP* **1009** (2010) 097 [1003.5827].
- [13] B. Harris, E. Laenen, L. Phaf, Z. Sullivan and S. Weinzierl, *The Fully differential single top quark cross-section in next to leading order QCD*, *Phys.Rev.* **D66** (2002) 054024 [hep-ph/0207055].
- [14] G. Bevilacqua, M. Czakon, C. Papadopoulos and M. Worek, *Dominant QCD Backgrounds in Higgs Boson Analyses at the LHC: A Study of $pp \rightarrow t \text{ anti-}t + 2 \text{ jets}$ at Next-To-Leading Order*, *Phys.Rev.Lett.* **104** (2010) 162002 [1002.4009].
- [15] S. Dittmaier, P. Uwer and S. Weinzierl, *NLO QCD corrections to $t \text{ anti-}t + \text{jet}$ production at hadron colliders*, *Phys.Rev.Lett.* **98** (2007) 262002 [hep-ph/0703120].
- [16] K. Melnikov and M. Schulze, *NLO QCD corrections to top quark pair production in association with one hard jet at hadron colliders*, *Nucl.Phys.* **B840** (2010) 129–159 [1004.3284].
- [17] G. Bevilacqua, M. Czakon, C. Papadopoulos, R. Pittau and M. Worek, *Assault on the NLO Wishlist: $pp \rightarrow t \text{ anti-}t b \text{ anti-}b$* , *JHEP* **0909** (2009) 109 [0907.4723].
- [18] A. Bredenstein, A. Denner, S. Dittmaier and S. Pozzorini, *NLO QCD Corrections to Top Anti-Top Bottom Anti-Bottom Production at the LHC: 2. full hadronic results*, *JHEP* **1003** (2010) 021 [1001.4006].
- [19] A. Denner, S. Dittmaier, S. Kallweit and S. Pozzorini, *NLO QCD corrections to off-shell top-antitop production with leptonic decays at hadron colliders*, 1207.5018.

- [20] K. Melnikov, A. Scharf and M. Schulze, *Top quark pair production in association with a jet: QCD corrections and jet radiation in top quark decays*, *Phys.Rev.* **D85** (2012) 054002 [1111.4991].
- [21] P. Baernreuther, M. Czakon and A. Mitov, *Percent level precision physics at the Tevatron: first genuine NNLO QCD corrections to $q\bar{q} \rightarrow t\bar{t} + X$* , 1204.5201.
- [22] M. Czakon and A. Mitov, *NNLO corrections to top-pair production at hadron colliders: the all-fermionic scattering channels*, 1207.0236.
- [23] C. Anastasiou and S. M. Aybat, *The One-loop gluon amplitude for heavy-quark production at NNLO*, *Phys.Rev.* **D78** (2008) 114006 [0809.1355].
- [24] B. Kniehl, Z. Merebashvili, J. Korner and M. Rogal, *Heavy quark pair production in gluon fusion at next-to-next-to-leading $O(\alpha_s^4)$ order: One-loop squared contributions*, *Phys.Rev.* **D78** (2008) 094013 [0809.3980].
- [25] J. Korner, Z. Merebashvili and M. Rogal, *NNLO $O(\alpha_s^4)$ results for heavy quark pair production in quark-antiquark collisions: The One-loop squared contributions*, *Phys.Rev.* **D77** (2008) 094011 [0802.0106].
- [26] M. Czakon, *Tops from Light Quarks: Full Mass Dependence at Two-Loops in QCD*, *Phys.Lett.* **B664** (2008) 307–314 [0803.1400].
- [27] R. Bonciani, A. Ferroglia, T. Gehrmann, D. Maitre and C. Studerus, *Two-Loop Fermionic Corrections to Heavy-Quark Pair Production: The Quark-Antiquark Channel*, *JHEP* **0807** (2008) 129 [0806.2301].
- [28] R. Bonciani, A. Ferroglia, T. Gehrmann and C. Studerus, *Two-Loop Planar Corrections to Heavy-Quark Pair Production in the Quark-Antiquark Channel*, *JHEP* **0908** (2009) 067 [0906.3671].
- [29] R. Bonciani, A. Ferroglia, T. Gehrmann, A. Manteuffel and C. Studerus, *Two-Loop Leading Color Corrections to Heavy-Quark Pair Production in the Gluon Fusion Channel*, *JHEP* **1101** (2011) 102 [1011.6661].
- [30] I. Bierenbaum, M. Czakon and A. Mitov, *The singular behavior of one-loop massive QCD amplitudes with one external soft gluon*, *Nucl.Phys.* **B856** (2012) 228–246 [1107.4384].
- [31] J. M. Campbell, M. Cullen and E. N. Glover, *Four jet event shapes in electron - positron annihilation*, *Eur.Phys.J.* **C9** (1999) 245–265 [hep-ph/9809429].
- [32] S. Catani and M. Seymour, *A General algorithm for calculating jet cross-sections in NLO QCD*, *Nucl.Phys.* **B485** (1997) 291–419 [hep-ph/9605323].
- [33] D. A. Kosower, *Antenna factorization of gauge theory amplitudes*, *Phys.Rev.* **D57** (1998) 5410–5416 [hep-ph/9710213].
- [34] Z. Kunszt and D. E. Soper, *Calculation of jet cross-sections in hadron collisions at order α_s^3* , *Phys.Rev.* **D46** (1992) 192–221.
- [35] G. Somogyi, *Subtraction with hadronic initial states at NLO: An NNLO-compatible scheme*, *JHEP* **0905** (2009) 016 [0903.1218].
- [36] U. Aglietti, V. Del Duca, C. Duhr, G. Somogyi and Z. Trocsanyi, *Analytic integration of real-virtual counterterms in NNLO jet cross sections. I.*, *JHEP* **0809** (2008) 107 [0807.0514].

- [37] P. Bolzoni, S.-O. Moch, G. Somogyi and Z. Trocsanyi, *Analytic integration of real-virtual counterterms in NNLO jet cross sections. II.*, *JHEP* **0908** (2009) 079 [0905.4390].
- [38] P. Bolzoni, G. Somogyi and Z. Trocsanyi, *A subtraction scheme for computing QCD jet cross sections at NNLO: integrating the iterated singly-unresolved subtraction terms*, *JHEP* **1101** (2011) 059 [1011.1909].
- [39] S. Catani and M. Grazzini, *An NNLO subtraction formalism in hadron collisions and its application to Higgs boson production at the LHC*, *Phys.Rev.Lett.* **98** (2007) 222002 [hep-ph/0703012].
- [40] M. Czakon, *A novel subtraction scheme for double-real radiation at NNLO*, *Phys.Lett.* **B693** (2010) 259–268 [1005.0274].
- [41] M. Czakon, *Double-real radiation in hadronic top quark pair production as a proof of a certain concept*, *Nucl.Phys.* **B849** (2011) 250–295 [1101.0642].
- [42] S. Frixione and M. Grazzini, *Subtraction at NNLO*, *JHEP* **0506** (2005) 010 [hep-ph/0411399].
- [43] W. B. Kilgore, *Subtraction terms for hadronic production processes at next-to-next-to-leading order*, *Phys.Rev.* **D70** (2004) 031501 [hep-ph/0403128].
- [44] D. A. Kosower, *Multiple singular emission in gauge theories*, *Phys.Rev.* **D67** (2003) 116003 [hep-ph/0212097].
- [45] G. Somogyi, Z. Trocsanyi and V. Del Duca, *Matching of singly- and doubly-unresolved limits of tree-level QCD squared matrix elements*, *JHEP* **0506** (2005) 024 [hep-ph/0502226].
- [46] G. Somogyi and Z. Trocsanyi, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Regularization of real-virtual emission*, *JHEP* **0701** (2007) 052 [hep-ph/0609043].
- [47] G. Somogyi and Z. Trocsanyi, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Integrating the subtraction terms. I.*, *JHEP* **0808** (2008) 042 [0807.0509].
- [48] S. Weinzierl, *Subtraction terms at NNLO*, *JHEP* **0303** (2003) 062 [hep-ph/0302180].
- [49] S. Catani, L. Cieri, G. Ferrera, D. de Florian and M. Grazzini, *Vector boson production at hadron colliders: A Fully exclusive QCD calculation at NNLO*, *Phys.Rev.Lett.* **103** (2009) 082001 [0903.2120].
- [50] S. Catani, G. Ferrera and M. Grazzini, *W Boson Production at Hadron Colliders: The Lepton Charge Asymmetry in NNLO QCD*, *JHEP* **1005** (2010) 006 [1002.3115].
- [51] M. Grazzini, *NNLO predictions for the Higgs boson signal in the $H \rightarrow WW \rightarrow l\nu l\nu$ and $H \rightarrow ZZ \rightarrow 4l$ decay channels*, *JHEP* **0802** (2008) 043 [0801.3232].
- [52] G. Ferrera, M. Grazzini and F. Tramontano, *Associated WH production at hadron colliders: a fully exclusive QCD calculation at NNLO*, *Phys.Rev.Lett.* **107** (2011) 152003 [1107.1164].
- [53] S. Catani, L. Cieri, D. de Florian, G. Ferrera and M. Grazzini, *Diphoton production at hadron colliders: a fully-differential QCD calculation at NNLO*, *Phys.Rev.Lett.* **108** (2012) 072001 [1110.2375].
- [54] T. Binoth and G. Heinrich, *An Automatized algorithm to compute infrared divergent multiloop integrals*, *Nucl.Phys.* **B585** (2000) 741–759 [hep-ph/0004013].
- [55] T. Binoth and G. Heinrich, *Numerical evaluation of multiloop integrals by sector decomposition*, *Nucl.Phys.* **B680** (2004) 375–388 [hep-ph/0305234].

- [56] G. Heinrich, *Sector Decomposition*, *Int.J.Mod.Phys.* **A23** (2008) 1457–1486 [0803.4177].
- [57] J. Carter and G. Heinrich, *SecDec: A general program for sector decomposition*, *Comput.Phys.Commun.* **182** (2011) 1566–1581 [1011.5493].
- [58] G. Heinrich, *A Numerical method for NNLO calculations*, *Nucl.Phys.Proc.Suppl.* **116** (2003) 368–372 [hep-ph/0211144].
- [59] C. Anastasiou, K. Melnikov and F. Petriello, *A New method for real radiation at NNLO*, *Phys.Rev.* **D69** (2004) 076010 [hep-ph/0311311].
- [60] T. Binoth and G. Heinrich, *Numerical evaluation of phase space integrals by sector decomposition*, *Nucl.Phys.* **B693** (2004) 134–148 [hep-ph/0402265].
- [61] G. Heinrich, *The Sector decomposition approach to real radiation at NNLO*, *Nucl.Phys.Proc.Suppl.* **157** (2006) 43–47 [hep-ph/0601232].
- [62] C. Anastasiou, K. Melnikov and F. Petriello, *Higgs boson production at hadron colliders: Differential cross sections through next-to-next-to-leading order*, *Phys.Rev.Lett.* **93** (2004) 262002 [hep-ph/0409088].
- [63] C. Anastasiou, G. Dissertori and F. Stockli, *NNLO QCD predictions for the $H \rightarrow WW \rightarrow l\nu l\nu$ signal at the LHC*, *JHEP* **0709** (2007) 018 [0707.2373].
- [64] C. Anastasiou, F. Herzog and A. Lazopoulos, *On the factorization of overlapping singularities at NNLO*, *JHEP* **1103** (2011) 038 [1011.4867].
- [65] S. Buehler, F. Herzog, A. Lazopoulos and R. Mueller, *The Fully differential hadronic production of a Higgs boson via bottom quark fusion at NNLO*, *Journal of High Energy Physics, Volume 2012, Number 7* (2012) , 115 [1204.4415].
- [66] K. Melnikov and F. Petriello, *Electroweak gauge boson production at hadron colliders through $O(\alpha_s^2)$* , *Phys.Rev.* **D74** (2006) 114017 [hep-ph/0609070].
- [67] A. Gehrmann-De Ridder, T. Gehrmann and E. N. Glover, *Antenna subtraction at NNLO*, *JHEP* **0509** (2005) 056 [hep-ph/0505111].
- [68] E. Nigel Glover and J. Pires, *Antenna subtraction for gluon scattering at NNLO*, *JHEP* **1006** (2010) 096 [1003.2824].
- [69] G. Abelof and A. Gehrmann-De Ridder, *Double real radiation corrections to $t\bar{t}$ production at the LHC: the all-fermion processes*, *JHEP* **1204** (2012) 076 [1112.4736].
- [70] A. Gehrmann-De Ridder, T. Gehrmann and E. N. Glover, *Quark-gluon antenna functions from neutralino decay*, *Phys.Lett.* **B612** (2005) 36–48 [hep-ph/0501291].
- [71] A. Gehrmann-De Ridder, T. Gehrmann and E. N. Glover, *Gluon-gluon antenna functions from Higgs boson decay*, *Phys.Lett.* **B612** (2005) 49–60 [hep-ph/0502110].
- [72] A. Daleo, T. Gehrmann and D. Maitre, *Antenna subtraction with hadronic initial states*, *JHEP* **0704** (2007) 016 [hep-ph/0612257].
- [73] A. Daleo, A. Gehrmann-De Ridder, T. Gehrmann and G. Luisoni, *Antenna subtraction at NNLO with hadronic initial states: initial-final configurations*, *JHEP* **1001** (2010) 118 [0912.0374].
- [74] T. Gehrmann and P. F. Monni, *Antenna subtraction at NNLO with hadronic initial states: real-virtual initial-initial configurations*, *JHEP* **1112** (2011) 049 [1107.4037].

- [75] A. Gehrmann-De Ridder, E. Glover and J. Pires, *Real-Virtual corrections for gluon scattering at NNLO*, *JHEP* **1202** (2012) 141 [[1112.3613](#)].
- [76] R. Boughezal, A. Gehrmann-De Ridder and M. Ritzmann, *Antenna subtraction at NNLO with hadronic initial states: double real radiation for initial-initial configurations with two quark flavours*, *JHEP* **1102** (2011) 098 [[1011.6631](#)].
- [77] A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, *Antenna subtraction at NNLO with hadronic initial states: double real initial-initial configurations*, [1207.5779](#).
- [78] G. Abelof and A. Gehrmann-De Ridder, *Antenna subtraction for the production of heavy particles at hadron colliders*, *JHEP* **1104** (2011) 063 [[1102.2443](#)].
- [79] W. Bernreuther, C. Bogner and O. Dekkers, *The real radiation antenna function for $S \rightarrow Q\bar{Q}q\bar{q}$ at NNLO QCD*, *JHEP* **1106** (2011) 032 [[1105.0530](#)].
- [80] A. Gehrmann-De Ridder and M. Ritzmann, *NLO Antenna Subtraction with Massive Fermions*, *JHEP* **0907** (2009) 041 [[0904.3297](#)].
- [81] S. Catani, S. Dittmaier and Z. Trocsanyi, *One loop singular behavior of QCD and SUSY QCD amplitudes with massive partons*, *Phys.Lett.* **B500** (2001) 149–160 [[hep-ph/0011222](#)].
- [82] S. Catani, S. Dittmaier, M. H. Seymour and Z. Trocsanyi, *The Dipole formalism for next-to-leading order QCD calculations with massive partons*, *Nucl.Phys.* **B627** (2002) 189–265 [[hep-ph/0201036](#)].
- [83] A. Gehrmann-De Ridder, M. Ritzmann and P. Skands, *Timelike Dipole-Antenna Showers with Massive Fermions*, *Phys.Rev.* **D85** (2012) 014013 [[1108.6172](#)].
- [84] W. T. Giele, D. A. Kosower and P. Z. Skands, *A Simple shower and matching algorithm*, *Phys.Rev.* **D78** (2008) 014026 [[0707.3652](#)].
- [85] W. Giele, D. Kosower and P. Skands, *Higher-Order Corrections to Timelike Jets*, *Phys.Rev.* **D84** (2011) 054003 [[1102.2126](#)].
- [86] J. Lopez-Villarejo and P. Skands, *Efficient Matrix-Element Matching with Sector Showers*, *JHEP* **1111** (2011) 150 [[1109.3608](#)].
- [87] A. Gehrmann-De Ridder, T. Gehrmann and E. N. Glover, *Infrared structure of $e^+e^- \rightarrow 2$ jets at NNLO*, *Nucl.Phys.* **B691** (2004) 195–222 [[hep-ph/0403057](#)].
- [88] G. Abelof, A. Gehrmann-De Ridder and O. Dekkers, *In Preparation*, .
- [89] J. M. Campbell and E. N. Glover, *Double unresolved approximations to multiparton scattering amplitudes*, *Nucl.Phys.* **B527** (1998) 264–288 [[hep-ph/9710255](#)].
- [90] S. Catani and M. Grazzini, *Collinear factorization and splitting functions for next-to-next-to-leading order QCD calculations*, *Phys.Lett.* **B446** (1999) 143–152 [[hep-ph/9810389](#)].
- [91] S. Catani and M. Grazzini, *Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond*, *Nucl.Phys.* **B570** (2000) 287–325 [[hep-ph/9908523](#)].
- [92] D. de Florian and M. Grazzini, *The Structure of large logarithmic corrections at small transverse momentum in hadronic collisions*, *Nucl.Phys.* **B616** (2001) 247–285 [[hep-ph/0108273](#)].

- [93] G. Altarelli and G. Parisi, *Asymptotic Freedom in Parton Language*, *Nucl.Phys.* **B126** (1977) 298.
- [94] A. Gehrmann-De Ridder, T. Gehrmann, E. Glover and G. Heinrich, *Infrared structure of $e^+e^- \rightarrow 3 \text{ jets}$ at NNLO*, *JHEP* **0711** (2007) 058 [[0710.0346](#)].
- [95] S. Weinzierl, *NNLO corrections to 2-jet observables in electron-positron annihilation*, *Phys.Rev.* **D74** (2006) 014020 [[hep-ph/0606008](#)].
- [96] R. Kleiss, W. J. Stirling and S. Ellis, *A new Monte Carlo treatment of multiparticle phase space at high-energies*, *Comput.Phys.Commun.* **40** (1986) 359.